Applied Physics 150b: Homework #2

(Dated: February 2, 2004)

Due: February 11, 2005 (in class).

1. **Reading**

   Chapter 1-3 from Allen and Eberly[1].

2. **(10 points) Bloch Vector Picture: conserved quantities and quantum-classical relations**

   (a) In class we derived the Bloch vector dynamics by defining three real quantities ($\{r_1, r_2, r_3\}$) in terms of the complex coefficients of the ground and excited states, and using the Schrödinger equation of the 2-level Rabi problem with electric dipole interaction. We also introduced the Pauli spin matrices, $\hat{\sigma}_i$, and related the expectation value of these operators to the Bloch vector components. Derive the equation of motion for the Bloch vector using the Heisenberg equations of motion for the $\hat{\sigma}_i$.

   (b) In light of the time-dependence of the $\hat{\sigma}_i$ in the Heisenberg picture derived above, show that certain conservation principles apply. In particular, show that $\hat{\sigma}_i(t)^2$ =Identity for all time (why are the commutation relations amongst the $\hat{\sigma}_i$ obviously fixed?). Also show that $\sum_i \langle\hat{\sigma}_i\rangle^2 = 1$ for all time. This last relation indicates that the length of the Bloch vector is conserved as it evolves in time, a characteristic that was obvious when we wrote the "gyroscopic" equation of motion for $\mathbf{R}$.

   (c) Assume Ω_R to be real, and consider the Bloch vector equations of motion. Discuss why r_1 can be thought of as the component (in units of the dipole transition element) of the atomic dipole moment *in-phase* (dispersive) with the driving electric field, and $(-)r_2$ as the *in-quadrature* (absorptive) component. If Ω_R were not real could we make this same designation? ***Hint***: Consider what drives transitions from the ground to excited state.

   (d) Assume that $r_3(t) = -1$ (i.e., atom is approximately in the ground-state (weak excitation)). Derive an approximate equation of motion for the expectation of the *slowly varying* atomic dipole moment amplitude. Compare your derived result with that of the Lorentz equation for a (lossless) *classical* dipole harmonic oscillator, and comment on the similarities and differences. It would seem that the farther from $r_3(t) \approx -1$, the less classical the atom behaves. Why then, is the classical theory successful under so many experimental conditions (i.e., classical dispersion theory of light)? ***Hint***: You may need to consider the effects of dissipation on the semi-classical equations (i.e., spontaneous emission). You may also want to consider the conservation laws relating to ($\{r_1, r_2, r_3\}$) in the quantum (semi-classical) case, and what effect $r_3(t) \approx -1$ has upon the normalized atomic dipole amplitude.

3. **(10 points) Pulse Area in the Bloch Vector Picture**

For this part of the problem, assume that $\Omega_R(t)$ is *real*, but time dependent, and that the phase and polarization of the applied laser field are held constant (i.e., $\mathbf{E}_l(\mathbf{r},t) = \hat{\varepsilon}\mathbf{E}_0(t)\cos(\omega_l t + \phi)$).

(a) Assuming that the atom can be represented by a pure state, show that the *zero-detuning* ($\delta = 0$) solution to the Bloch equations in a rotating reference frame ("primed" basis as defined in class) are:

$$r_1'(t) = r_1'(-\infty), \tag{1}$$

$$r_2'(t) = [1 - r_1'^2]^{1/2} \sin[\Theta(t) + K], \tag{2}$$

$$r_3'(t) = [1 - r_1'^2]^{1/2} \cos[\Theta(t) + K], \tag{3}$$

where the "tipping angle", $\Theta(t)$, is related to the time-dependent Rabi-flopping frequency by,

$$\dot{\Theta}(t) = \Omega_R(t). \tag{4}$$

K is a time independent phase constant related to the intitial values of $r_2'(-\infty)$ and $r_3'(-\infty)$. Solve for these relations and rewrite eqs. (1-3) in terms of $r_2'(-\infty)$ and $r_3'(-\infty)$.

(b) The solution found above in (a), represents oscillation in the $2' - 3'$ plane, with $\Theta(t)$ the angle of clockwise rotation in this plane (hence the name "tipping angle"). Show that the total angle of rotation is related only to the *area* under the envelope of an applied laser pulse.

(c) Assume we had used phase modulation of the laser field as opposed to amplitude modulation (i.e, $\phi(t)$ was taken as time-dependent, and $\mathbf{E}_0$ held fixed). Reformulate the Bloch equations in the rotating reference frame, keeping the phase terms.

(d) Continuing with part (c), assume that time is broken up into units of $\pi/2$ pulse area for the given fixed Ω_R (i.e., a *digital* phase-modulation scheme). If the atom were to start in the ground state, and then one were to apply three consecutive "bits" equal to, $\phi(t) = (\pi/2, 0, \pi/2)$, what final state would the atom be left in? Show all work related to obtaining this answer. Assume that the transition between "bit" values of $\phi(t)$ is extremely rapid, not allowing the atomic system to respond. ***Bonus***: Discuss how rapid "exteremely rapid" must be, and what effects a limited slew-rate on the digital phase-modulation might have on this modulation scheme.

For the remaining parts to this problem we now take the detuning to be *non-zero* ($\delta \neq 0$), and only allow amplitude modulation of the laser light (take the phase, $\phi = 0$).

(e) If we take the initial conditions of the atom to be, $r_1'(-\infty) = r_2'(-\infty) = 0$ and $r_3'(-\infty) = -1$, show that the solution to eqs. (1-3) yields, $r_3'(t) = -1 + 2\sin^2[\frac{1}{2}\Theta(t)]$. In other words, in the case of zero-detuning, we get periodic motion of the atomic state, with the atom returning to the ground state after each "2π"-pulse. Note that you may think this a trivial result in light of our previous discussion in class of "Rabi-flopping", but in that case we assumed Ω_R a fixed quantity. Here we see that for zero-detuning we get a similar behaviour, needing only to integrate the effects of $\Omega_R(t)$.

(f) Assume that in the *non-zero* detuning case we can write the motion of the atomic inversion as,

$$r_3'(t) = -1 + 2B\sin^2[\frac{1}{2}\Theta(t)], \tag{5}$$

where B is a parameter that is less than one (reduced amplitude of oscillation) for $\delta \neq 0$. This solution is analogous to the behaviour of the atom in the case of fixed Ω_R, where we saw that the "Rabi-flops" were reduced in amplitude and occured with a modified (higher) frequency. Since we are dealing now with a non-fixed $\Omega_R(t)$, this *ansatz* puts a constraint on the type of pulses that will yield such periodic motion. Show that in order to have the atomic inversion satisfy the periodic motion described in eq. (5), $\Theta(t)$ must satisfy the following relation,

$$\dot{\Theta}(t) = -2\delta\left(\frac{B}{1-B}\right)^{1/2}\sin[\frac{1}{2}\Theta(t)]. \tag{6}$$

This equation determines the shape of the laser pulse to be applied to the atom.

(g) Let's further assume that the shape of the laser pulse should not depend upon detuning (we will see why below). The coefficient in front of $\sin[\frac{1}{2}\Theta(t)]$ in eq. (6) should then be independent of the detuning frequency and have units of inverse time. Defining $1/T = |\delta|(B/1-B)^{1/2}$, show that

$$\Theta(t) = 4\arctan\left[\exp\left(\frac{t-t_o}{T}\right)\right], \tag{7}$$

where t_o is the (arbitrary) time where $\Theta = \pi$ (i.e., the center of the pulse). Show that this gives for the laser pulse shape,

$$\Omega(t) = \frac{2}{T}\mathrm{sech}\left(\frac{t-t_o}{T}\right). \tag{8}$$

(h) With $\Omega(t)$ given by eq. (8), solve for the corresponding time evolution of $r'_1(t)$, $r'_2(t)$, and $r'_3(t)$. Show that, regardless of the atoms detuning, a laser pulse of amplitude and hyperbolic secant shape as given by eq. (8) will result in the atom returning to the ground state at the end of the pulse. As a result, no energy is absorbed from the laser pulse, regardless of the fact that the laser is on- or near-resonance where the atomic media is expected to be highly absorbing. This is the basis of *self-induced transparency*, first discovered by McCall and Hahn in 1967.

4. **(10 points) Half-Integer Spin Fermionic Oscillator**

It can be shown[2] that in order to have a stable ground state of a half-integer spin field (Fermionic field) that the canonical commutation relations for the creation and annihilation operators (or equivalently the field and its conjugate momentum) must be replaced by anti-commutation relations. This fundamentally changes the spectrum for the Fermionic fields, and is responsible for the anti-symmetric behavior of multi-particle Fermionic wave functions and the Pauli-exclusion principle.

(a) Using the same form of the Hamiltonian as for the Simple Harmonic Oscillator (SHO)

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \tag{9}$$

find the spectrum of a Fermionic oscillator with the commutator brackets ($[]$) replaced by anti-commutator brackets ($\{\}$): $\{\hat{a},\hat{a}^\dagger\} = 1$ and $\{\hat{a},\hat{a}\} = \{\hat{a}^\dagger,\hat{a}^\dagger\} = 0$. *Hint*: Follow the same procedure you might use for deriving the spectrum of a (Bosonic) SHO.

(b) From (a) what are the number states which span the Fermionic oscillator Hilbert space? Comment on your findings with regard to the Pauli-exclusion principle.

(c) Solve for the time evolution of the Fermionic $\hat{a}$ and $\hat{a}^\dagger$ in the Heisenberg picture.

(d) Consider a system of two independent Fermionic oscillators with Hamiltonian

$$\hat{H} = \sum_{i=1,2} \hbar\omega(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}), \tag{10}$$

where $\{\hat{a}_i,\hat{a}_j^\dagger\} = \delta_{ij}$ and $\{\hat{a}_i,\hat{a}_j\} = \{\hat{a}_i^\dagger,\hat{a}_j^\dagger\} = 0$ (i.e., independent creation or annihilation operators anti-commute now instead of commuting). Using your result from (b), what are the Fock space kets which span the Hilbert space? Argue that the Fock space kets are anti-symmetric with respect to oscillator 1 and oscillator 2. *Hint*: Consider the relation between $\hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle$ and $\hat{a}_2^\dagger\hat{a}_1^\dagger|0\rangle$.

(e) Consider a state of the 2 particle Fermionic system in (d),

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|n_1 = 0\rangle \otimes |n_2 = 1\rangle + |n_1 = 1\rangle \otimes |n_2 = 0\rangle). \tag{11}$$

What is the density operator matrix, $\hat{\rho}$, for this state in the Fock space basis?

(f) Imagine that you only had local access to oscillator 1, and as such could only measure the state of oscillator 1. Given that the state of the *entire* system is given by $|\Psi\rangle$, what would be the density operator matrix, $\hat{\rho}_1$, for the oscillator 1 subspace (the effective *system* for those local to oscillator 1).

5. **(10 points) The Density Operator in Thermodynamic Equilibrium: Quantum Statistical Mechanics**

   For a system in thermodynamic equilibrium with temperature T the density operator is diagonal in the energy eigenstate basis[3]:

$$\hat{\rho} = Z^{-1} e^{-\beta\hat{H}}, \tag{12}$$

where $\beta = 1/k_B T$, k_B is Boltzmann's constant, $\hat{H}$ is the Hamiltonian operator for the system, and Z is the partition function which is chosen such that $\mathrm{Tr}(\hat{\rho}) = 1$.

(a) Show that

$$Z = \sum_n g_n e^{-\beta E_n}, \tag{13}$$

where g_n is the degeneracy of the nth energy level with energy E_n.

(b) Consider a system consisting of a single SHO with Hamiltonian, $\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + 1/2)$, and the canonical commutation relations between $\hat{a}$ and $\hat{a}^\dagger$. Calculate Z for this system. Show that the density operator of eq. (12) results in Bose statistics for the average occupation level of the SHO by calculating the expectation value of the number operator, $\hat{n} = \hat{a}^\dagger \hat{a}$.

(c) Repeat (b) for a Fermionic oscillator (with canonical anti-commutation relations). Show that one gets Fermi statistics for the average occupation level of the oscillator.

Note: Bose statistics for a single Bosonic oscillator, of frequency ω, predicts that the average number state occupied by an oscillator in thermal equilibrium with its environment at absolute temperature T is

$$\overline{n}_b = \langle \hat{n} \rangle_b = \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \text{ (Bose)}, \tag{14}$$

For a single Fermionic oscillator

$$\overline{n}_f = \langle \hat{n} \rangle_f = \frac{e^{-\beta\hbar\omega}}{1 + e^{-\beta\hbar\omega}} \text{ (Fermi)}. \tag{15}$$

---

[1] L. Allen and J. Eberly, *Optical Resonance and Two-Level Atoms* (Dover Publications, Inc., Mineola, NY, 1987).

[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, New York, NY, 1997).

[3] J. J. Sakurai, *Modern Quantum Mechanics* (Springer, New York, NY, 1994), revised ed.