Applied Physics 190c: Homework #3

(Dated: May 13, 2019)

Due: Tuesday, May 21, anytime.

I. READING

Chapter 1 in “Statistical Methods of Quantum Optics” [Carmichael, 1999] (see notes on class website). Chapter 4 in “Methods in Theoretical Quantum Optics” [Barnett and Radmore, 1997], Chapter 3 and 4 in “Statistical Methods of Quantum Optics” [Carmichael, 1999].

II. (50 POINTS) DRIVEN CAVITY-QED SYSTEM: MORE PRACTICE WITH THE QUANTUM OPTICS TOOLBOX

A. Hamiltonian in the rotating-wave approximation

Write down the Hamiltonian for a system consisting of a two-level atom (transition frequency $\omega_a$) coupled to a single-mode cavity (resonance frequency $\omega_c$) with vacuum coupling rate $g$. Assume you have an optical drive (of amplitude $E$) which couples to the electric field of the cavity mode at frequency $\omega_L$. Make a rotating wave approximation (RWA), and write your Hamiltonian in a frame rotating with your input laser drive frequency.

B. Liouvillian in the Born-Markov limit

Assume that you have both atomic decay via spontaneous emission at rate $\gamma$ and cavity energy decay via radiation at rate $\kappa$. Write down a Liouvillian valid in the Born-Markov limit. In deriving the Liouvillian you may assume that the electromagnetic bath is in the vacuum state and that radiation modes that the cavity leaks into are distinct from the radiation modes that the atom spontaneously emits into.

C. Weak coupling limit

For cavity tuned exactly on resonance with the atom ($\omega_c = \omega_a$), and for parameters $g = 0.1$, $\kappa = \gamma = 0.2$, and $E = 0.01$, use the Quantum Optics Toolbox (or Qu-Tip) to calculate and plot the steady-state emission intensity (photon count rate) of the atom and the cavity versus the drive frequency $\omega_L$ for optical drives near resonance. Provide your code, and comment on the observed spectrum.

D. Strong coupling limit

Repeat the calculations in (c) and plot for an atom-cavity system in the strong coupling limit, $g = 1 > \kappa, \gamma$. Comment on the observed spectrum.

E. Detuned response

Repeat the calculations in (d) for a series of atom-cavity detunings. Comment on the observed spectra.
F. Atomic saturation

Repeat the calculations in (e) for a range of optical drive amplitudes, \( E = 0.01, 0.05, 0.1, 0.5 \). Comment on and explain the trend observed in the spectra.

III. (50 POINTS) ATOMIC RESONANCE FLUORESCENCE VIA QME: THE QUANTUM REGRESSION THEOREM IN PRACTICE

Consider a two-level atom in free-space with lower energy (ground) electronic state \( |g\rangle \) and upper energy (excited) electronic state \( |e\rangle \), in which you drive the atomic transition near resonance with a narrow-linewidth laser (all other modes of free-space are assumed to be in vacuum). Here we will be interested in the properties of the total optical field which results due to the near-resonant scattering of the applied laser light by a single atom. We will find that the atom produces both elastic scattering (i.e., no frequency shift) and inelastic scattering of the applied laser. The scattered laser light will also have interesting highly non-classical intensity correlations. Note, you will want to consult Carmichael’s book \( \text{[Carmichael, 1999]} \) for this problem, although be sure to show intermediate steps in calculations!

A. Quantum Master Equation (QME)

Show that the Linblad form of the QME for such a system is given by \( \hbar = 1 \),

\[
\dot{\hat{\rho}} = -i \left[ \left( \Delta \hat{s}_z - \frac{1}{2}(\Omega \hat{s}_+ + \Omega^* \hat{s}_-) \right), \hat{\rho} \right] - \frac{\gamma}{2} (\hat{s}_+ \hat{s}_- \hat{\rho} - 2\hat{s}_- \hat{\rho} \hat{s}_+ + \hat{\rho} \hat{s}_+ \hat{s}_-),
\]

where \( \hat{\rho} \) is the density operator of the atomic two-level system (radiation field traced out), \( \hat{s}_+ = |e\rangle \langle g| \) (atomic raising operator), \( \hat{s}_- = |g\rangle \langle e| \) (atomic lowering operator), \( \hat{s}_z = (1/2)(|e\rangle \langle e| - |g\rangle \langle g|) \) (atomic inversion operator), \( \Omega \) is the complex Rabi frequency which is proportional to the product of the laser’s electric field amplitude at the position of the atom and the atomic transition electric dipole moment, and \( \gamma \) is the atomic spontaneous emission rate in free-space (i.e., into the continuum of free-space radiation modes). Here we have chose to work in a frame rotating at the frequency of the laser drive frequency \( (\omega_l) \) such that \( \Delta = (\omega_a - \omega_l) \) is the detuning of the laser from the atomic transition frequency \( (\omega_a) \). Note that you need not rigorously derive or justify the QME, just intuit it from the Hamiltonian of the driven atom system or make an analogy to the example of a damped harmonic oscillator coupled to a thermal bath considered in class, where \( \hat{a} \to \hat{s}_- \) and \( \hat{a}^\dagger \to \hat{s}_+ \) and here we take the bath in vacuum \( (\bar{n} = 0) \).

B. Single-time expectations and the evolution matrix

Using the QME, derive the equations of motion for the time-dependent expectation values of the atomic lowering, raising, and inversion operators: \( \partial / \partial t \left\langle \hat{s} \right\rangle = \textbf{M} \left\langle \hat{s} \right\rangle + \textbf{b} \), where (in the Heisenberg picture) \( \hat{s}(t) \equiv (\hat{s}_-(t), \hat{s}_+(t), \hat{s}_z(t)) \), \( \textbf{M} \) is a \( 3 \times 3 \) region of the evolution matrix and \( \textbf{b} \) is a constant vector. Solve for the steady-state expectation values of the atomic operators, \( \left\langle \hat{s} \right\rangle_{ss} \).

C. Spectrum of a driven atom

The spectrum of the light field scattered by a driven atom is related to the Fourier transform of the two-time correlation of the atomic raising and lowering operators, \( G^{(1)}(t, t + \tau) \approx \left\langle \hat{s}_+(t) \hat{s}_-(t + \tau) \right\rangle \). Specifically, the steady-state spectrum is given by,
\[ S_{ss}[\omega] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \lim_{t\to\infty} \langle \hat{s}_+(t) \hat{s}_-(t + \tau) \rangle. \] (2)

Note, that here we assume that we are detecting the atomic spectrum at some off-angle from the laser drive so that we only measure the scattered field. By separating the fluctuating and average values of the atomic operators, \( \hat{s}(t) = \langle \hat{s} \rangle + \delta \hat{s}(t) \), solve for the steady-state, elastic component of the scattered field spectrum (hint: you should find your solutions for \( \langle \hat{s} \rangle \) valuable here). Comment on the scaling of this elastic component of the scattered field with laser drive power. Does it make physical sense to you?

D. Inelastic component of the spectrum and the QRT

You should find in your analysis of the atomic resonance fluorescence spectrum from the previous section that the inelastic component (in steady-state) of the spectrum depends upon the two-time correlator \( \langle \delta \hat{s}_+(t) \delta \hat{s}_-(t + \tau) \rangle_{ss} \). The Quantum Regression Theorem allows us to solve for this two-time correlator using the evolution matrix \( \mathbf{M} \) found from the single-time expectations values, \( \langle \delta \hat{s}_+(t) \delta \hat{s}_-(t + \tau) \rangle_{ss} = \exp[\mathbf{M}\tau]\langle \delta \hat{s}_+(t) \delta \hat{s}(t) \rangle_{ss} \). Solve for the eigenvalues and the corresponding steady-state values of the eigenvectors of \( \mathbf{M} \) at zero detuning (\( \Delta = 0 \)). Using the similarity transformation that diagonalizes \( \mathbf{M} \), solve for \( \hat{s}_- \) in terms of the eigenvectors of \( \mathbf{M} \). With these results in hand, plot the steady-state, inelastic component of the atomic resonance fluorescence under limits of weak (\( |\Omega|^2 \ll (\gamma/2)^2 \)) and strong (\( |\Omega|^2 \gg (\gamma/2)^2 \)) laser driving. You should find the emergence of two sidebands in the spectrum, at \( \omega \pm \Omega \). Comment on the physical nature of these sidebands (hint: consider the "dressing" of the atom by the laser due to rapid absorption and re-emission of the laser drive).

E. Absorption spectrum: Bonus (20 points)

The inelastic fluorescence spectrum under strong drive has a triplet of resonance peaks, the so-called "Mollow triplet". We measure this spectrum by measuring free-space radiation from the atom off-axis from the laser drive. If one were to measure the transmitted laser field (i.e., on-axis) as a function of laser detuning, what sort of spectrum would we observe? Derive this "absorption" spectrum using the QME. Come to terms with why the fluorescence and absorption spectra look so different under strong laser driving. Explain your thoughts.

F. Intensity correlations

The intensity correlations of the atomic resonance fluorescence can be calculated in steady-state using,

\[ G^{(2)}_{ss}(t, t + \tau) = \lim_{t \to \infty} \langle \hat{s}_+(t) (\hat{s}_+(t + \tau) \hat{s}_-(t + \tau)) \hat{s}_-(t) \rangle = \lim_{t \to \infty} \langle \hat{s}_+(t) \left( \hat{s}_+(t + \tau) + \frac{1}{2} \right) \hat{s}_-(t) \rangle. \] (3)

Physically, we should expect these intensity correlations to be given by, \( G^{(2)}_{ss}(t, t + \tau) = [\text{probability of the atom emitting a 1st photon}] \times [\text{probability that a second photon is emitted at time } \tau \text{ after first photon, conditioned on the atom starting in the ground state}] \). Mathematically this looks like,

\[ G^{(2)}_{ss}(t, t + \tau) = [\langle \hat{s}_+ \hat{s}_- \rangle] \times [\langle \hat{s}_+(\tau) \hat{s}_-(\tau) \rangle |_{\rho(0) = \rho_{ss}} = |g \rangle \langle g |] , \] (4)

where we have used the fact that the probability of emitting a photon is proportional to the excited state probability, \( P_e = \langle |e \rangle \langle e | \rangle = \langle \hat{s}_+ \hat{s}_- \rangle \). Use the QRT to derive the relation for \( G^{(2)}_{ss}(t, t + \tau) \) given in Eq. (4).
G. $g^{(2)}(\tau)$ curves: Bonus (5 points)

For a single bonus point, what are the $\tau = 0$ and $\tau = \infty$ limits of $g^{(2)}(\tau)$? Using the QME and the equations of motion for $\langle \hat{s} \rangle (t)$ previously derived, solve for $\langle \hat{s}(\tau) \rangle_{\hat{\rho}(0) = |g\rangle\langle g|}$ using an initial condition of the atom $\hat{\rho}(0) = |g\rangle\langle g|$, and plot the normalized intensity fluctuations, $g^{(2)}_{ss}(t,t+\tau) \equiv (\lim_{\tau \to \infty} G^{(2)}_{ss}(t,t+\tau))^{-1} G^{(2)}_{ss}(t,t+\tau)$, as a function of inter-photon separation $\tau$ in the limits of weak and strong driving of the atom on resonance ($\Delta = 0$). Comment on the results and try to provide some physical intuition for the two curves.

H. $g^{(2)}(\tau)$ for filtered resonance fluorescence: Bonus (5 points)

Think about what might happen if you were to measure not the total scattered field of the driven atom, but rather a filtered version of the fluorescence. In particular, if you were to filter for the central Mollow triplet, or one of the sideband peaks, do the photon statistics change? What about cross-correlations in the emitted peaks?

REFERENCES