Applied Physics 150a: Homework #2

(Dated: November 4, 2014)

Due: Thursday, November 13th, anytime before midnight. There will be an "INBOX" outside my office in Watson (Rm. 266/268).

1. **(20 points) Hong-Ou-Mandel (HOM) interferometer revisited**

In class we discussed the case of a pair of indistinguishable photons colliding on a 50:50 beam splitter. For a lossless 50:50 beam splitter (represented by a unitary matrix), we found that for two photons (one at each input port of the beam splitter) there appeared a perfect interference between both photons transmitting through the beam splitter and both photons reflecting off of the beam splitter. Both of these "paths" produce a photon in each output port and they are the only two paths that do so. As such the probability of obtaining a photon in each output port is zero. The only other possibilities are that the photons exit together, both in the same output port. For this reason, photons which are bosons, are said to "bunch" together. Our class discussion did not touch on how the fact that photons are bosons plays a role in this bunching. We shall do so here.

(a) Let's label the input ports of the beam splitter as a and b (the output ports will be a' and b'), and take the quantum state of a particle entering port a as $|a\rangle$. Think of the input and output quantum states as representing the plane wave with appropriate direction to enter or exit the appropriate beam splitter port. We will also take without loss of generality the unitary matrix representing the beam splitter as, $\hat{U} = ir|a'\rangle\langle a| + t|a'\rangle\langle b| + ir|b'\rangle\langle b| + t|b'\rangle\langle a|$:

$$\hat{U} = \begin{pmatrix} ir & t \\ t & ir \end{pmatrix}, \tag{1}$$

where $\begin{pmatrix} a' \\ b' \end{pmatrix} = \hat{U}\begin{pmatrix} a \\ b \end{pmatrix}$. Here $\hat{U}$ represents the transformation for a single particle passing through the beam splitter (the particle can enter in a superposition of each input port state and exit in a superposition of output port states. Show for a symmetric input state, $|\psi\rangle_s = 1/\sqrt{2}(|a\rangle_1|b\rangle_2 + |b\rangle_1|a\rangle_2)$, that passage through the 50:50 beam splitter results in both particles exiting together in one of the two output ports. Here I have explicitly labeled each state with a subscript to keep track of the individual particles, and the input state is symmetric with respect to exchange of the particles (their labels). Not also that $\hat{U}$ acts on the different particles separately in each product state of particles. Bosons in quantum mechanics must be symmetrized under particle exchange as in the symmetric input state $|\psi\rangle_s$, and so this represents the bosonic photon scattering we considered in class.

(b) What is the symmetry of the output state obtained in part (a)? You will also recall in class that we obtained a probability for the two photons exiting together in each port which was a factor of two too small (the total probability summed to 1/2). With the new properly symmetrized input and output states obtained from a more careful accounting of each particle's passage, calculate the probability for the two photons to leave in port a' and b' (now pat yourself on the back for not being off by a factor of two).

(c) If one considers a pair of indistinguishable electrons (fermions) colliding on a 50:50 beam splitter, the result is completely opposite to the photon case. The fermionic electrons always end up one in each output port of the beam splitter, and never both in the same output port. Prove this by starting with an anti-symmetric input state.

(d) In a classic experiment by Hong, Ou, and Mandel [1], they measured the collision of a pair of "identical" photons generated through type-I parametric down-conversion in a nonlinear crystal. Read the paper, and sketch out their experimental set-up indicating the purpose of each of the components. What determines the width of the HOM dip in their measurement? What do you think would happen to the second-order coherence curve if identical electrons were collided on the 50:50 beam splitter?

(e) Bonus (5 points): You will notice that the analysis above assumed only a spatial part to the wave function of the particles. This is natural as the beam splitter only performs spatial mapping between different modes of the particles. Real electrons and photons also have a spin degree of freedom (polarization for the light field) that the beam splitter doesn't discriminate against. Come up with a unitary matrix for single particle scattering across a 50:50 beam splitter for photons with horizontal (H) and vertical (V) polarizations.

2. **(20 points) Creation and annihilation operators**

To begin, find a good textbook [2], and review the properties of a quantum simple harmonic oscillator. In particular, find a derivation which focuses on the energy eigenbasis, and raising and lowering operators between the different energy levels.

The above analysis of the Hong-Ou-Mandel interferometer involving two photon interactions at a beam splitter utilized a labeling scheme to keep track of different "particles" and symmetrization of these multi-particle states to account for the bosonic or fermionic nature of the particles involved. The book keeping is admittedly tedious. There is a somewhat more straight forward approach to dealing with multi-particle states in quantum mechanics that emerges naturally from a quantum field theory. A quantum field theory can be constructed from a classical field theory by relating the eigenmodes of the classical field theory with energy eigenstates of the quantum field theory (QFT).

A simple example would be the classical electromagnetic field in free space. The eigenmodes of harmonic time dependence are plane waves, $e^{i(\mathbf{k}\cdot\vec{r}-\omega_{\mathbf{k}}t)}$, where the harmonic angular frequency is related to the magnitude of the wave vector through the speed of light c as $\omega_{\mathbf{k}} = c|\mathbf{k}|$. An expansion of the total field in terms of appropriately normalized classical field eigenmodes, then yields coefficients of the expansion which we upgrade to quantum operators. Again for our example free space electromagnetic field, this might look like $\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}} \left(\hat{b}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\vec{r}-\omega_{\mathbf{k}}t)} + \hat{b}^{\dagger}_{\mathbf{k}} e^{-i(\mathbf{k}\cdot\vec{r}-\omega_{\mathbf{k}}t)} \right)$, where $b_{\mathbf{k}}$ ($b^{\dagger}_{\mathbf{k}}$) is the annihilation (creation) operator for the plane wave eigenmode with wave vector $\mathbf{k}$ and $\mathbf{E}_{\mathbf{k}}$ is a normalization factor (see below). Note that in order to get a real field in the end we need both positive and negative time harmonics, and we have chosen to associate positive time harmonics with "creation" and negative time harmonics with "annihilation" of our eigenmodes.

Since these operators involve time harmonic modes, their equations of motion look identical to that of the raising ($\hat{a}^{\dagger}$) and lowering ($\hat{a}$) operators of a quantum simple harmonic oscillator (QSHO). With this analogy in mind, one then simply equates the QFT creation operator with the raising operator of the QSHO and the QFT annihilation operator with the QSHO lowering operator. This whole process is called "second quantization", in recognition of the fact that first quantization involves coming up with a field theory description for a single particle (think Schroedinger's equation for an electron), and then a second quantization is needed to quantize the amplitude of the single particle field theory using the rules described above.

(a) If we take the QSHO analogy seriously, what should the commutation relations between $b_{\mathbf{k}}$ and $b^{\dagger}_{\mathbf{k}}$ be set to? [A natural question then arises about what to do with operators of different eigenmodes (different $\mathbf{k}$ in our example). The answer is very subtle and very deep.

The different eigenmodes are to be treated as different particle states, and thus the operators should commute as if they were independent simple harmonic oscillators.]

(b) For a homogeneous material of dielectric constant ε and finite volume V, find the appropriate normalization $\mathbf{E_k}$ constant such that the creation and annihilation operators operate on fields with a quantum of energy given by $\hbar\omega_{\mathbf{k}}$. Hint: all you need to know is how to calculate the energy of a classical electromagnetic field.

(c) Our description so far only works for bosonic systems. This is easily seen by the fact that each eigenmode corresponds to a different particle state, and operation with the creation operator of that state just increases the QSHO level by one (i.e., adds one more particle). For a QSHO one can keep operating with the raising operator and raise the system to the next energy (particle) level, without any limit. Clearly, this violates the Pauli exclusion principle for fermions which forbids two particles occupying the same quantum state. In order to address this there is a very simple fix. Instead of using commutation relations one assumes anti-commutation relations for a fermionic field theory: $\{f, f^\dagger\} \equiv \left(f + f^\dagger\right)$. Replacing every commutation bracket with an anti-commutation curly bracket in the relations you found in (a), show that a fermionic field theory satisfies the Pauli exclusion principle.

(d) Show using the commutation relations of a bosonic field and the anti-commutation relations of a fermionic field, that bosons are symmetric in terms of particle rearrangement and fermions are anti-symmetric. Hint: you should think of particle rearrangement in this case as the order in which you add a particle to the field.

(e) Go back and redo the Hong-Ou-Mandel interferometer problem with colliding bosons using your new field theory operators. In this case, each port is associated with a different eigenmode of the field (a different direction or $\mathbf{k}$), and so a particle coming into port a would be represented by $b_a^\dagger|0\rangle$. What is the input state corresponding to one boson entering port a and one boson entering port b? The unitary transformation (matrix) that relates the input ports to the output ports of a beam splitter, should now be used to relate the operators at the input ports to the operators at the output ports. Using this "input-output" formalism you should then be able to write the input state in terms of output operators operating on the vacuum. Using the boson commutation relations you should find that because of cancellations in the terms of products of operators, you get only two-particle states exiting one or the other output port, and never one in each. Find the probability of two bosons exiting one of the output ports (hint: you must assume the normalization properties of the raising operator of the QSHO). Repeat this for two colliding fermions. Note how much easier and less confusing the particle book keeping is when using the creation operators.

3. **(20 points) Unconditional Quantum Teleportation of Continuous Variable Quantum States**

   The first demonstration of quantum teleportation discussed in class were for an arbitrary single qubit state represented in the polarization of a photon [3]. Shortly after this first demonstration, Jeff Kimble and his group demonstrated a form of unconditional quantum teleportation of the continuous variables associated with an infinite-dimensional Hilbert space. Carefully read the paper by Furusawa, et al., [4], and answer the following questions as best you can.

   (a) What physical form does the input state which is to be teleported take, and what are the degrees of freedom that describe that state?

   (b) What are the entangled states used in this teleportation protocol and how are they generated in practice?

   (c) What is the Bell-State measurement that Alice performs on the input state and her half of the entangled state she shares with Bob? How did the Kimble group do this in practice?

   (d) Alice then communicates (classically) the result of her Bell-State measurement to Bob, who must use this information to recreate the input state from his half of the shared entangled state. Mathematically, what is the transformation that Bob must apply to his half of the entangled state. Under what conditions of the entangled state will he be able to perfectly reproduce the unknown input state? What would the fidelity of teleportation be if Alice and Bob had used a pair of unsqueezed photons (regular coherent state photons) as their shared resource for teleportation?

   (e) Bonus (5 points): The claim in Ref. [4] is that not only are they teleporting a continuous variable quantum state, but they are also doing so unconditionally. In the first experiment of Bouwmeester, et al. [3], where is the conditioning being done?

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[1] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987).

[2] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics: Volume I* (Wiley-Interscience: a division of John Wiley & Sons, Inc., New York, NY, 1977).

[3] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eible, H. Weinfurter, and A. Zeilinger, Nature **390**, 575 (1997).

[4] A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).