# Applied Physics 150a: Homework #3

(Dated: November 13, 2014)

**Due:** Thursday, November 20th, anytime before midnight. There will be an "INBOX" outside my office in Watson (Rm. 266/268).

1. **(10 points) The Interaction Picture(s)**

   (a) In class we saw that for some unitary transformation $\hat{U}$ on the state vector, the effective Hamiltonian (the operator which generates time translation on the transformed state vector) transforms as $\hat{H}_{\mathrm{I}} = i\hbar\dot{\hat{U}}\hat{U}^\dagger + \hat{U}\hat{H}_S\hat{U}^\dagger$, where $\hat{H}_S$ is the Schrödinger picture Hamiltonian. Show that for a Schrödinger Hamiltoninan $\hat{H}_S = \hat{H}_0 + \hat{H}'(t)$, with $\hat{H}_0$ any time-independent component of the Hamiltonian, then for the judicious choice of $\hat{U} = \exp(i\hat{H}_0 t/\hbar)$ we arrive at an interaction picture Hamiltonian of $\hat{H}_{\mathrm{I}} = \exp(i\hat{H}_0 t/\hbar)\hat{H}'(t)\exp(-i\hat{H}_0 t/\hbar)$.

   (b) The semiclassical atom-field Hamiltonian for a 2-level atom interacting with a classical laser field (with Rotating Wave Approximation (RWA) already applied) is given by,

$$\hat{H} = \hbar\omega_e|e\rangle\langle e| + \hbar\omega_g|g\rangle\langle g| + \left(\frac{\hbar\Omega_R e^{-i\omega_l t}}{2}|e\rangle\langle g| + c.c\right). \tag{1}$$

   Here, $\omega_l$ is the frequency of the applied laser field, $\omega_a \equiv \omega_e - \omega_g$ is the atomic transition frequency, $\delta \equiv \omega_l - \omega_a$, and $\Omega_R$ is the bare Rabi-flopping frequency. In class we derove the quantum version of this Hamiltonian for a single mode of the optical field. What is the relation for $\Omega_R$ in terms of the classical field intensity and dipole moment of the 2-level atom? Show that under unitary transformation $\hat{U} = \exp\{i(\omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g|)t\}$ the interaction picture Hamiltonian is $\hat{H}_{\mathrm{I}} = \left(\frac{\hbar\Omega_R e^{-i\delta t}}{2}|e\rangle\langle g| + c.c\right)$.

   (c) Show that the interaction Hamiltonian in the case of $\hat{U} = \exp\{i((\omega_g + \omega_l)|e\rangle\langle e| + \omega_g|g\rangle\langle g|)t\}$ is the time-independent, $H_{\mathrm{I}} = -\hbar\delta|e\rangle\langle e| + \left(\frac{\hbar\Omega_R}{2}|e\rangle\langle g| + c.c\right)$.

   (d) In all of the interaction picture studies thus far in class we have implicitly been working in the *Schrödinger* interaction picture. There is another interaction picture, the *Heisenberg* interaction picture, which is also useful in many cases, and widely used in quantum optics. Show that for an operator with no explicit time dependence that the appropriate equation of motion of a Heisenberg interaction picture operator is,

$$\dot{\hat{A}}_{I,H} = \frac{i}{\hbar}\left([\hat{H}_{\rm I}, \hat{A}_I]\right)_H . \tag{2}$$

Note the explicit use of subscripts here. A subscript $I$ indicates the *Schrödinger* interaction picture. The additional subscript $H$ indicates the Heisenberg picture.

2. **(20 points) The degenerate parametric amplifier**

   In nonlinear optics, difference frequency generation annihilates a photon at frequency $\omega_1$ and creates photons at frequencies $\omega_2$ and $\omega_3$ such that $\omega_1 = \omega_2 + \omega_3$. When the frequencies are such that $\omega_1 = 2\omega$ and $\omega_2 = \omega_3 = \omega$, the process is called degenerate parametric amplification, because a low frequency input at $\omega$ can be amplified by the high frequency pump at $2\omega$.

   In the following, assume that the pump beam is intense enough that it can be treated classically. The input signal mode at frequency $\omega$ has annihilation operator $\hat{a}$, and the Hamiltonian for this process can then be written as (***bonus***: derive this Hamiltonian (physically motivate each term) for extra credit):

$$H = \hbar\omega\hat{a}^\dagger\hat{a} - \frac{i\hbar\xi}{2}(\hat{a}^2 e^{2i\omega t} - (\hat{a}^\dagger)^2 e^{-2i\omega t}) \tag{3}$$

   where $\xi$ is a nonlinear constant related to the $\chi^{(2)}$, the second-order nonlinear susceptibility.

   (a) Show that in the interaction picture, this Hamiltonian can be written as:

$$H_{\rm I} = -\frac{i\hbar\xi}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2) \tag{4}$$

   To show this, it might be helpful to use the following general identity, known as the operator expansion theorem (Mandel and Wolf [1], Section 10.11.1):

$$\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B} + x[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \ldots \tag{5}$$

   (b) We can define two quadrature phase amplitude operators by $\hat{X}_1 = \hat{a} + \hat{a}^\dagger$ and $\hat{X}_2 = -i(\hat{a} - \hat{a}^\dagger)$. Solve the Heisenberg equations of motion for these two operators, assuming that the initial values for the operators are $\hat{X}_1(0)$ and $\hat{X}_2(0)$.

(c) Solve for the variances of the two quadrature phase amplitude operators in terms of their variances at time $t = 0$. If the field is initially in a coherent state, what are the variances of these operators? Discuss the physical meaning of these results.

(d) The normalized second-order correlation function of *photodetected* light at position $\mathbf{r}$ (i.e., normal ordered operator correlations), for a statistically non-stationary field, can be written as ($\tau \geq 0$):

$$g^2(\mathbf{r};t,\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle} \tag{6}$$

Assuming an initial vacuum state at $t = 0$, calculate $g^2(\mathbf{r};t,0)$. Next, assume an initial coherent state, and calculate $g^2(\mathbf{r};t,0)$ (I recommend looking up established mathematical identities for coherent state operations [1]). What are the physical implications of these results?

3. **(30 points) Reversible single photon gun**

Recently there has been a general interest for quantum information systems in the creation of light sources which produce single photon pulses (producing one, and only one photon in a pulse is in fact non-trivial due to the Bosonic character of light). For many applications, these single photon "guns" must also have other, more subtle attributes. They typically need to generate the photon "on-demand" (i.e., not stochastically), reversibly for the reading and writing of quantum information within a quantum network [2], and with high efficiency in proposed *efficient* quantum computation protocols with linear optics (LOQC) [3].

To date, only atomic cavity QED (cQED) systems have been able to produce such novel states of light. The schemes used thus far rely upon the multi-level character and coherence of the atomic system, and fall under the umbrella of processes involving Stimulated Raman scattering with Adiabatic Passage (STIRAP). A schematic of a typical cQED single-photon generation process is shown in Figure 1. In the cQED photon guns, one arm of the Raman transition is supplied by a classical coherent laser source (intensity denoted by the resonant Rabi-flopping frequency $\Omega_{\mathrm{T}}$), the other by the quantum field within a small mode volume, high-quality-factor (Q) cavity (resonant vacuum Rabi-flopping frequency $= 2g$). The dressed state analysis studied in class provides one of the most clear views of how STIRAP processes work and of the cQED single-photon pulse sources that utilize them.

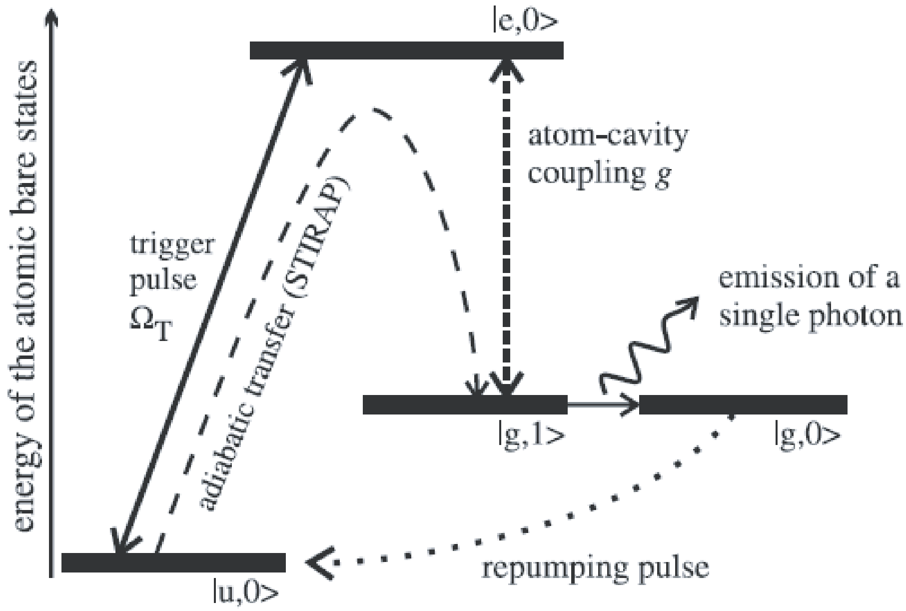


FIG. 1: Schematic representation of a STIRAP process with the strongly coupled vacuum field of a cavity acting as one arm of the Raman transition.

(a) If we define $\Delta$ as the common detuning between the classical laser source and the cavity mode from the excited state $|e,0\rangle$ (i.e., $\Delta_C = \omega_C - (\omega_e - \omega_g) = \Delta_T = \omega_T - (\omega_e - \omega_u) = \Delta$), then show that the interaction picture Hamiltonian can be written as (assuming RWA):

$$\hat{H}_{\mathrm{I}} = -\frac{\hbar}{2}\begin{pmatrix} 0 & \Omega_{\mathrm{T}} & 0 \\ \Omega_{\mathrm{T}}^{*} & -2\Delta & 2g \\ 0 & 2g^{*} & 0 \end{pmatrix}, \tag{7}$$

in the uncoupled basis of $\{|u,0\rangle, |e,0\rangle, |g,1\rangle\}$. Note that the states are written with the first parameter corresponding to the atomic state and the second parameter corresponding to the Fock state of the cavity (quasi-) mode ($\Omega_{\mathrm{T}}$ is classical, and as such does not show up in the quantum state definition). The state $|g,0\rangle$, reached after the photon is emitted from the cavity, is not included in the above basis as it is non-resonant in the dressed-state picture with the other states.

(b) Assuming $\Omega_{\mathrm{T}}$ and $g$ real (and time-independent for the moment), show that the dressed eigenstates of this system are

$$|a^0\rangle = \cos\Theta|u,0\rangle - \sin\Theta|g,1\rangle, \tag{8}$$

$$|a^+\rangle = \cos\Phi\sin\Theta|u,0\rangle - \sin\Phi|e,0\rangle + \cos\Phi\cos\Theta|g,1\rangle, \tag{9}$$

$$|a^-\rangle = \sin\Phi\sin\Theta|u,0\rangle + \cos\Phi|e,0\rangle + \sin\Phi\cos\Theta|g,1\rangle, \tag{10}$$

$$\tag{11}$$

where the mixing angles $\Theta$ and $\Phi$ are given by

$$\tan\Theta = \frac{\Omega_\mathrm{T}}{2g}, \tag{12}$$

$$\tan\Phi = \frac{\sqrt{4g^2+\Omega_\mathrm{T}^2}}{\sqrt{4g^2+\Omega_\mathrm{T}^2+\Delta^2}+\Delta}. \tag{13}$$

$$\tag{14}$$

Also find the corresponding eigenvalues.

(c) From the results found in (b), explain how one might use state $|a^0\rangle$ to "trigger" the generation of a single photon in the cavity mode starting from an un-excited atom with an empty cavity. *Hint*: Think about *adiabatically* adjusting some of the system parameters.

(d) Find an adiabatic limit for the photon generation rate of your scheme in (c). Use some realisitic assumptions in your model to provide a simple answer in terms of $g$ and $\Omega_\mathrm{T}$. Note that this is not the *only* limit on the photon generation rate as the cavity decay rate ($\kappa$) and the atomic excited state decay rate ($\gamma$) are also important. ***Bonus:*** Argue what relative values $g$, $\kappa$, and $\gamma$ should take in order to most efficienty generate single photons into the cavity mode.

(e) If the Raman transition was not a $\Lambda$-transition (common excited state), but rather a $V$-transition with common ground-state of the atom, could we still make a "triggered" single photon pulse source? If so, what would be the "trigger" source in this case. If not, why not. *Hint*: It will help to think about what possible uncoupled atom-cavity states are nearly resonant in this case.

[1] L. Mandel and E. Wolf, *Optical coherence and quantum optics* (Cambridge University Press, New York, NY, 1995).

[2] J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, Phys. Rev. Lett. **78**, 3221 (1997).

[3] E. Knill, R. Lalamme, and G. J. Milburn, Nature **409**, 46 (2001).