Applied Physics 150a: Final Homework #4

(Dated: December 10, 2014)

Due: Friday, December 12th, anytime before midnight. There will be an "INBOX" outside my office in Watson (Rm. 266/268).

1. **Recommended Reading:** Michel Devoret's Les Houches notes on quantum fluctuations in electrical circuits [1], Steve Girvin's Les Houches notes on circuit QED [2], John Martinis' Les Houches notes on superconducting quantum circuits [3], the original Transmon paper [4], and some of the early Josephson Junction papers [5, 6]. Links to these papers can be found on the class website.

2. **(20 points) The Quantum LC Resonator**

   (a) In class we presented the Hamiltonian for a resonant *LC* circuit in terms of the flux through an inductor and the charge on a capacitor. In these notes we assumed that the charge on the capacitor (Q) was the canonical "momentum" and the flux through the inductor (Φ) the canonical "position". In this case what is the canonical commutation relation between charge and flux? What are Hamilton's equations of motion? Draw out the LC circuit showing the direction of current through the inductor and the sign of the charge on capacitor. Be sure to check that your direction and sign convention are consistent with Hamilton's equations. **Note:** One can also take the charge on the capacitor to be the canonical "position". One arrives at similar equations, just with a different sign choice for the charge on the capacitor.

   (b) In analogy to the mechanical simple harmonic oscillator, what is the relation between "mass" and "spring constant" and capacitance and inductance of the circuit? From the resonant frequency of a mechanical oscillator then, what is the resonant frequency of the circuit in terms of L and C? Using this formal analogy to the SHO, write out the corresponding raising ($\hat{a}^\dagger$; creation) and lowering ($\hat{a}$; annihilation) operators for the circuit in terms of the charge operator $\hat{Q}$ and the flux operator $\hat{\Phi}$ of the circuit.

   (c) Again using this analogy, what are the zero-point fluctuations in the charge (Q_{zpf}) and the flux (Φ_{zpf}) of the circuit? Write these fluctuations in terms of the characteristic impedance of the circuit, defined as $Z = (L/C)^{1/2}$. Recall from class we defined the *superconducting* *flux quantum* as $\Phi_0 \equiv h/(2e)$, where h is Planck's constant. One can similarly define a *superconducting resistance quantum*, $R_Q = h/(2e)^2$. Write the zero-point fluctuations for charge and flux in terms of only the electronic charge (e), the flux quantum Φ_0, and a normalized impedance $z \equiv Z/R_Q$.

(d) Write out the charge ($\hat{Q}$) and flux ($\hat{\Phi}$) operators in terms of Q_{zpf}, Φ_{zpf} and the raising and lowering operators.

(e) The voltage across the inductor should be equal to the time rate of change of the flux winding through the inductor. From the operator Heisenberg equation of motion and the canonical commutation relation, derive the voltage operator $\hat{V}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. What is the zero-point fluctuation amplitude of the voltage, V_{zpf}, in terms of circuit parameters? What is the relation between V_{zpf} and Φ_{zpf}?

(f) What is Φ_0 in units of microVolts per GHz? How about R_Q in Ohms? For a circuit with resonant frequency of 10 GHz, characteristic impedance of 1 kOhm what is the size of the voltage fluctuations across the capacitor if the circuit is in its quantum ground-state?

3. **(40 points) From the Cooper Pair Box to the Transmon**

In class we discussed both the Cooper Pair Box (CPB) and the Transmon superconducting quantum circuit elements which are formed using a Josephson Junction (JJ). Due to the strong nonlinearity of the JJ inductance with superconducting quantum phase across the junction, one can obtain under the right external circuit parameters a resonant circuit which is both anharmonic at the single excitation level *and* insensitive to external charge fluctuations. Here we will review the important properties of the CPB and Transmon.

(a) Draw a circuit representation of a CPB, indicating the charge "box" or "island". What are the two macroscopic degrees of freedom of the CPB?

(b) What are the Josephson relations between the current through and voltage across a JJ and the phase difference $\delta \equiv \phi_1 - \phi_2$ between the superconducting state in the two superconducting films defining the junction?

(c) From the Josephson relations derive the effective Josephson inductance, L_J, of the junction. Derive a relation for the (classical) energy stored in this inductive part of the junction circuit, and use this to motivate a Hamiltonian for the CPB which also includes the charging energy of the "box" due to Cooper pairs tunneling across the junction *and* any external potential that may be applied (a "gate" effect). Define a tunneling energy E_J and a charging energy E_C.

(d) Let's consider the phenomenological "hopping" Hamiltonian,

$$\hat{H}_T \equiv -\frac{E_J}{2}\sum_N (|N+1\rangle\langle N| + |N\rangle\langle N+1|), \tag{1}$$

as a representation of the microscopic tunneling of Cooper pairs across the JJ, where state $|N\rangle$ represents the integer N number of excess Cooper pairs in the "box". Let's also define a set of phase states $|\delta\rangle$ which are the Fourier dual to the number states $|N\rangle$, $|\delta\rangle \equiv \sum_N \exp iN\delta|N\rangle$ (Note that the phase δ lives on the compact space $[0, 2\pi]$ as the Cooper pair number N on the "island" is a discrete integer.). Show that the states $|\delta\rangle$ are eigenstates of $\hat{H}_T$ with eigenvalue $\lambda_{T,\delta} = -E_J\cos(\delta)$. If one thinks of $|\delta\rangle$ as a plane wave with wavevector δ, then the group velocity of this state is given by $v_g \equiv (1/\hbar)\partial\lambda_{T,\delta}/\partial\delta$. Show that the current associated with this state is the same as that given by the (first) Josephson relations, where the state $|\delta\rangle$ is considered to be a state with well defined superconducting phase δ across the JJ.

(e) The number operator for the number of excess Cooper pairs in the box is $\hat{N} \equiv \sum_N N|N\rangle\langle N|$. Show that $\hat{N}|\delta\rangle = -i\partial/\partial\delta(|\delta\rangle)$, in direct analogy to the position and momentum operator of a SHO.

(f) Show how one can use a pair of JJs, forming a SQUID loop [6], to create a CPB with a tunable tunneling energy E_J.

(g) Draw a circuit which captures the main components of a Transmon, and comment on the importance of each of the main components to the functionality of the Transmon qubit (include read-out and E_J tuning).

(h) Draw an energy band diagram for the Transmon in the limit $E_J = 0$ and in the limit $E_J \sim E_C$. What is the charge dispersion of the Transmon in the limit $E_J/E_C \gg 1$? See Ref. [4].

(i) Following Ref. [4], expand the $\cos(\hat{\delta})$ operator of the CPB Hamiltonian to fourth order in $\hat{\delta}$, to arrive at an approximate Duffing Hamiltonian for the CPB. Considering only the SHO part of the resulting Hamiltonian, define new raising and lowering operators for these harmonic levels. What are the charge and phase zero-point fluctuations of the SHO in this limit? Based upon the phase zero-point fluctuations, does it make sense to have restricted ourselves to small values of phase near zero?

(j) Again referring to Ref. [4], what are the approximate eigen-energies of the Transmon in the limit $E_J/E_C \gg 1$. To leading order in perturbation theory, what harmonic oscillator states are mixed to form the energy eigenstates?

(k) In the absence of charge dispersion in the Transmon, a microwave cavity can be used to control and read-out the qubit states. The interaction between the Transmon and a capacitively coupled microwave resonator is proportional to the charge on the island, $\hat{N}$. Since $\hat{N}$ is not a good quantum number of the bare Transmon, the microwave cavity tends to couple the different energy eigenstates. What is the Cooper Pair number operator of the Transmon in terms of the SHO creation and annihilation operators in the limit $E_J/E_C \gg 1$? Use this along with your answer in part (j) to show that the microwave cavity will selectively couple Transmon states different by only one excitation level ($|j\rangle \Leftrightarrow |j \pm 1\rangle$). Derive an approximate form for the matrix element $\left|\langle j+k|\hat{N}|j\rangle\right|$ for $k = 1$ and $k > 1$.

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[1] M. H. Devoret, Les Houches Lectures (1997).

[2] S. M. Girvin, Les Houches Lectures (2011).

[3] J. M. Martinis and K. Osborne, Les Houches Lectures (2011).

[4] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **76**, 042319 (2007).

[5] B. D. Josephson, Physics Letters **1**, 251 (1962).

[6] R. C. Jaklevic, J. Lambe, A. H. Silver, and J. E. Mercereau, Phys. Rev. Lett. **12**, 159 (1964).