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Optomechanical creation of magnetic fields for photons on a lattice: supplementary material

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1. DERIVATION OF THE EFFECTIVE MAGNETIC HAMILTONIAN FOR THE MODULATED LINK SCHEME

Here, we derive the effective Hamiltonian $\hbar J_{\rm eff} e^{i\phi}\hat{a}_B^\dagger \hat{a}_A + \text{h.c.}$ that describes the tunneling of photons from site A to site B in the presence of an effective magnetic field created using the modulated link scheme. We start from the full time dependent Hamiltonian for the modulated link scheme (as displayed in the main text). Since this second-quantized Hamiltonian is particle conserving we can switch to a first-quantized picture in the standard way. The corresponding single-particle Hamiltonian $\hat{H}_M$ reads

$$\hat{H}_M = \hbar \begin{pmatrix} \omega_A & 0 & -J \\ 0 & \omega_B & -J \\ -J & -J & \bar{\omega}_I + 2g_0|\beta|\cos(\Omega t+\phi) \end{pmatrix}.$$

It acts on the photon wavefunction $|\psi\rangle \equiv (\psi_A, \psi_B, \psi_I)$ where ψ_s describes the probability amplitude that the photon is localized on site s, $s = A, B, I$. Since the Hamiltonian is time periodic, there is a complete set of quasi-periodic solutions of the Schrödinger equation, $|\psi_j(t + 2\pi/\Omega)\rangle = \exp[-i2\pi\omega_j/\Omega]|\psi_j(t)\rangle$ where $2\pi/\Omega$ is the period and index j spans the Hilbert space, $j = 1, 2, 3$. In practice, one solves the eigenvalue problem $\varepsilon_{jm}|\phi_{jm}\rangle\rangle = \mathcal{H}_{jm}|\phi_{jm}\rangle\rangle$ where $\mathcal{H} \equiv -i\hbar\partial_t + \hat{H}_M$ is the Floquet-Hamiltonian, $\varepsilon_{jm} = \hbar(\omega_j + m\Omega)$ are the quasienergies and $|\phi_{jm}\rangle\rangle = \exp[i(\omega_j + m\Omega)t]|\psi_j(t)\rangle$ are time-periodic states, the so-called Floquet eigenstates [$m \in \mathbb{Z}$] [1]. Notice that the Floquet Hamiltonian can be regarded as an operator on the extended Hilbert space of the time periodic vectors equipped with the scalar product

$$\langle\langle\phi_j|\phi_m\rangle\rangle = \frac{1}{T}\int_0^T \langle\phi_j(t)|\phi_m(t)\rangle. \qquad \textbf{(S1)}$$

In this framework, we can use the standard quantum mechanical perturbation theory to derive an effective time independent single-particle Hamiltonian. We assume a resonant drive $\omega_A \approx \omega_B + \Omega$, and weak tunneling/driving, $J_{A,B}/|\omega_{A,B} - \bar{\omega}_I|$, $g_0\beta/\Omega \ll 1$. We identify resonant Floquet-levels with quasienergies $\hbar\omega_A$ and $\hbar(\omega_B + \Omega)$ coupled via the third order virtual tunneling process through the interface site I. Up to leading order in perturbation theory, we can focus on the block of the Floquet Hamiltonian comprising the four unperturbed quasienergy levels that are involved in this process,

$$\hat{\mathcal{H}} = \hbar \begin{pmatrix} \omega_A & 0 & -J & 0 \\ 0 & \omega_B+\Omega & 0 & -J \\ -J & 0 & \omega_I & g_0\beta \\ 0 & -J & g_0\beta^* & \omega_I+\Omega \end{pmatrix}.$$

Application of a standard Schrieffer-Wolff transformation [2–4], i.e. applying degenerate perturbation theory to third order, leads to the effective block diagonal Floquet Hamiltonian

$$\hat{\mathcal{H}}_{\rm eff} = \hbar \begin{pmatrix} \tilde{\omega}_A & J_{\rm eff}e^{-i\phi} \\ J_{\rm eff}e^{i\phi} & \tilde{\omega}_B + \omega_{\rm ex} \end{pmatrix}. \qquad \textbf{(S2)}$$

where $\tilde{\omega}_s = \omega_s + J_s^2/(\omega_s - \bar{\omega}_I)$ with $s = A, B$ and $J_{\rm eff} = g_0|\beta|J_A J_B/[(\omega_A - \bar{\omega}_I)(\omega_B - \bar{\omega}_I)]$. Finally, we turn back Hamiltonian S2 into its second-quantized form and switch to a frame rotating with frequency $\tilde{\omega}_A$ ($\tilde{\omega}_B$) on site A (B). For a resonant drive, $\tilde{\omega}_A = \tilde{\omega}_B + \Omega$, this yields the desired form of the second-quantized effective Hamiltonian, $J_{\rm eff}(e^{-i\phi}\hat{a}_A^\dagger \hat{a}_B + e^{i\phi}\hat{a}_B^\dagger \hat{a}_A)$.

2. TRANSMISSION AMPLITUDES AND DENSITY OF STATES FOR THE MODULATED LINK SCHEME

Here, we calculate the LDOS for the modulated link scheme which is plotted in Figure 2 of the main text. We use the full time dependent Hamiltonian for the modulated link scheme (displayed in the main text), extended to the whole lattice (including also the sublattice formed by the link sites). Since we are dealing with a time periodic system where the energy is not a constant of motion, we have to appropriately generalize the definition of the LDOS. A natural generalization of the standard definition to time-periodic systems is the following,

$$\rho(\omega, \mathbf{j}) = -2{\rm Im}G(\omega, 0; \mathbf{j}, \mathbf{j})$$

where $G(\omega, m; \mathbf{j}, \mathbf{l})$ is the Floquet Green's function

$$G = \frac{-i}{T}\int_0^T d\tau \int_0^\infty dt e^{i(\omega+m\Omega)t + im\Omega\tau}\langle[\hat{a}_\mathbf{j}(t+\tau), \hat{a}_\mathbf{l}^\dagger(\tau)]\rangle.$$

The Floquet Green's function describes the (linear) response of the array to a probe laser. More precisely, the light amplitude on site $\mathbf{j}$ in the presence of a probe drive on site $\mathbf{l}$ with frequency ω and amplitude $\alpha^{(in)}$ [described by the additional Hamiltonian term $H_I = i\hbar\sqrt{\kappa}\alpha^{(in)}(\hat{a}_\mathbf{l}^\dagger e^{-i\omega t} + h.c.)$] is

$$\langle \hat{a}_\mathbf{j}(t)\rangle = \sum_m e^{-i(\omega+m\Omega)t} i\sqrt{\kappa}\alpha^{(in)} G(\omega, m; \mathbf{j}, \mathbf{l}).$$

This is essentially a generalization of the Kubo formula which applies to any time periodic Hamiltonian. Using the input-output relations, $\hat{a}_\mathbf{j}^{(out)}(t) = \hat{a}_\mathbf{j}^{(in)}(t) - \sqrt{\kappa}\hat{a}_\mathbf{j}$, we can also calculate the field outside the cavity,

$$\langle \hat{a}_\mathbf{j}^{(out)}(t)\rangle \equiv \sum_m e^{-i(\omega+m\Omega)t} t_O(\omega, m; \mathbf{j}, \mathbf{l})\alpha^{(in)},$$

where

$$t_O(\omega, m; \mathbf{j}, \mathbf{l}) = \delta_{\mathbf{jl}}\delta_{m0} - i\kappa G(\omega, m; \mathbf{j}, \mathbf{l})$$

is the transmission amplitude of a photon from site $\mathbf{l}$ to site $\mathbf{j}$ if it has been up-converted m-times (or down-converted $|m|$-times for m negative).

For a time-periodic system with a particle conserving Hamiltonian, the Floquet Green's function can be easily expressed in terms of the first-quantized Floquet Hamiltonian $\mathcal{H} = -i\partial_t - H(t)$,

$$G(\omega, m; \mathbf{j}, \mathbf{l}) = \langle\langle \mathbf{j}, m | (\omega - \mathcal{H} + i\kappa/2)^{-1} | \mathbf{l}, 0\rangle\rangle$$

Notice that the Floquet Hamiltonian and the Green's function can be regarded as operators acting on the extended Hilbert space of the time-periodic photon states with the scalar product Eq. (S1). As such they acts on the time periodic states $|\mathbf{j}, m\rangle\rangle$, where index $\mathbf{j}$ indicates the lattice site and m the Fourier component. Thus, the density of states can be readily computed by diagonalizing the Floquet Green's function. We find

$$\rho(\omega) = \sum_k \frac{\kappa}{(\omega-\omega_k)^2 + \kappa^2/4}\left|\langle\langle \mathbf{j}, 0 | \phi_k\rangle\rangle\right|^2,$$

where $\hbar\omega_k$ are the quasienergies and $|\phi_k\rangle\rangle$ are the corresponding Floquet eigenstates obtained by numerically diagonalizing $\mathcal{H}$. Taking into account that the Floquet eigenfunctions $|\phi_k\rangle\rangle$ forms a complete orthonormal basis of the Hilbert space of the time-periodic states [with the scalar product Eq. (S1)], it immediately follows that the density of states is appropriately normalized,

$$\int_{-\infty}^\infty d\omega \rho(\omega) = 2\pi.$$

3. TRANSMISSION AMPLITUDES FOR THE FREQUENCY-CONVERSION SCHEME

For the frequency-conversion scheme we start from the linearized Langevin equations for the full array including the mechanical links modes [5, 6],

$$\begin{aligned} \dot{\hat{b}}_\mathbf{k} &= i\hbar^{-1}[\hat{H}, \hat{b}_\mathbf{k}] - \Gamma\hat{b}_\mathbf{k}/2 + \sqrt{\Gamma}\hat{b}_\mathbf{k}^{(\rm in)}, \\ \dot{\hat{a}}_\mathbf{j} &= i\hbar^{-1}[\hat{H}, \hat{a}_\mathbf{j}] - \kappa\hat{a}_\mathbf{j}/2 + \sqrt{\kappa}\hat{a}_\mathbf{j}^{(\rm in)}. \end{aligned} \quad \textbf{(S3)}$$

The first line (second line) describes the sites hosting a mechanical (optical) mode. The Hamiltonian $\hat{H}$ is given by the Hamiltonian for the wavelength conversion scheme (as displayed in the main text), extended to the full array, and the noise forces have the usual commutation relations [5]. Notice that Eq. (S3) is written in a frame where the optical modes on sublattice A and B are rotating with frequency ω_{L1} and ω_{L2}, respectively. A probe laser on site $\mathbf{l}$ with frequency ω and amplitude α^{in} is described by the additional Hamiltonian term $H_I = i\hbar\sqrt{\kappa}\alpha^{(in)}(\hat{a}_\mathbf{l}^\dagger e^{-i\Delta_p t} - h.c.)$, where $\Delta_p = \omega - \omega_{Ls}$ ($s = 1, 2$ for $\mathbf{l}$ on sublattice A or B, respectively). The linear response of the light amplitude on site $\mathbf{j}$ to such probe laser is given by the Kubo formula

$$\begin{aligned} \langle \hat{a}_\mathbf{j}(t)\rangle &= i\sqrt{\kappa}\alpha^{(in)} e^{-i\Delta_p t} G_{\hat{a}\hat{a}^\dagger}(\Delta_p, \mathbf{j}, \mathbf{l}) \\ &\quad - i\sqrt{\kappa}\alpha^{(in)} e^{i\Delta_p t} G_{\hat{a}\hat{a}}(-\Delta_p, \mathbf{j}, \mathbf{l}), \end{aligned} \quad \textbf{(S4)}$$

with the Green's functions

$$\begin{aligned} G_{\hat{a}\hat{a}^\dagger}(\omega, \mathbf{j}, \mathbf{l}) &= -i \int_0^\infty dt e^{i\omega t}\langle[\hat{a}_\mathbf{j}(t), \hat{a}_\mathbf{l}^\dagger(0)]\rangle, \\ G_{\hat{a}\hat{a}}(\omega, \mathbf{j}, \mathbf{l}) &= -i \int_0^\infty dt e^{i\omega t}\langle[\hat{a}_\mathbf{j}(t), \hat{a}_\mathbf{l}(0)]\rangle. \end{aligned}$$

Notice that in Figure 3 and 4 of the main text we plot the resonant part of the response corresponding to the first line of Eq. (S4). If $\mathbf{j}$ and $\mathbf{l}$ lie on different sublattices, the frequency of the probe signal is converted [to read off this frequency from Eq. (S4), one has to keep in mind that the frame of reference is rotating at different frequencies on the two optical sublattices]. Finally, we note that the light transmitted outside of the sample $\left\langle \hat{a}_\mathbf{j}^{(\rm out)}\right\rangle = t(\omega, \mathbf{j}, \mathbf{l})\left\langle \hat{a}_\mathbf{l}^{(\rm in)}\right\rangle$ can be readily computed using the input output relations [7] $\hat{a}_\mathbf{j}^{(\rm out)} = \hat{a}_\mathbf{j}^{(\rm in)} - \sqrt{\kappa}\hat{a}_\mathbf{j}$. From Eq. (S4) we find the transmission amplitude

$$t_O(\omega, \mathbf{l}, \mathbf{j}) = \delta_{\mathbf{lj}} - i\kappa G_{\hat{a}\hat{a}^\dagger}(\omega, \mathbf{l}, \mathbf{j}). \quad \textbf{(S5)}$$

Since the transmission amplitudes of a probe laser beam are *generally* proportional to the corresponding light amplitudes inside the array (on all sites except for the one where the light is injected), the amplitude patterns shown in Figures 3 and 4 could be directly measured by a position resolved measurement of the light scattered by the array.

In order to calculate the transmission in Figures 3 and 4 we have calculated the Green's function numerically. We note that for an array with $N \times N$ optical sites, there is a total of $N(2N-1)$ sites (including also the mechanical sites) and a total of $2N(2N-1)$ degrees of freedom. Thus, computing numerically the Green's function amounts to inverting a $2N(2N-1) \times 2N(2N-1)$ matrix. In Figure 3 and 4 we have chosen $N = 22$.

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