Adaptive Phase Measurements of Optical Modes: Going Beyond the Marginal Q Distribution

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In standard single-shot measurements of the phase of an optical mode, the phase and amplitude quadratures are jointly measured, and the latter information discarded. These techniques are consequently suboptimal. Here I suggest an adaptive scheme, whereby the phase is estimated from the results so far and fed back to control the phase of the local oscillator so as to measure the (estimated) phase quadrature only. I show that adaptive phase measurements can approach optimal phase measurements for states with both low and high mean photon numbers.

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The phase ϕ of a single mode of the electromagnetic field is the quantity canonically conjugate to the photon number n. It is now generally accepted that there exists a unique *canonical* probability distribution function (PDF) for this variable [1]:

$$P_{\mathrm{can}}(\phi) = \mathrm{Tr}[\rho F_{\mathrm{can}}(\phi)], \quad (1)$$

where $F_{\mathrm{can}}(\phi)$ is a positive-operator-valued measure (POVM) [1–3] for ϕ defined in terms of the unnormalized phase states $|\phi\rangle$:

$$F_{\mathrm{can}}(\phi) = \frac{1}{2\pi}|\phi\rangle\langle\phi|, \qquad |\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle. \quad (2)$$

This PDF is guaranteed to be normalized from the requirement on all POVMs that $\int d\lambda\, F(\lambda) = 1$, where λ is the measurement result. Considerable work has been done showing how this distribution can be inferred from physically realizable homodyne measurements on an arbitrarily large ensemble of identical copies of the system [4]. However, this ability is not at all the same as the ability to make canonical phase measurements. To do the latter, one would have to make a measurement on a *single copy* of the system, the result of which would be a random variable drawn from the canonical PDF (1). There is no known way to achieve this in general, nor is there ever likely to be.

One practical reason for wishing to make canonical phase measurements is for efficient communication [2]. If one encoded information in the phase of a single-mode optical pulse (which is easy to do with an electro-optic modulator), then one would wish the receiver to measure that phase as accurately as possible. In a canonical phase measurement the error in the measured phase would be limited only by the intrinsic quantum uncertainty in the phase [2]. Therefore it is only if a receiver could make a canonical (or near to canonical) phase measurement that schemes for preparing states which have minimum intrinsic phase uncertainty [4] would be able to be fully exploited for efficient communication.

At present, there are a number of practical (noncanonical) schemes for single-shot phase measurements, all of which give equivalent results [1]. One of these schemes (which here stands in place of any of them) is heterodyne detection, which uses a local oscillator highly detuned from the system. The two Fourier components of the photocurrent record yield measurements of both quadratures of the field [5]. These can be converted into results for the intensity and phase, the former of which is discarded. Because half of the measurement information is useless, this phase measurement is far from canonical. The POVM for such standard measurements is

$$F_{\mathrm{std}}(\phi) = \int_0^{\infty} \tfrac{1}{2}\, dn\, F_{\mathrm{het}}(\phi, n). \quad (3)$$

Here the heterodyne POVM $F_{\mathrm{het}}(\phi, n)$ is defined in terms of coherent states of complex amplitude $A = \sqrt{n}\, e^{i\phi}$:

$$F_{\mathrm{het}}(\phi, n) = \pi^{-1}|A\rangle\langle A|, \quad (4)$$

where the result A (defined later) encodes both Fourier amplitudes. In other words, the PDF for standard phase measurements is the marginal phase PDF of the Q function $Q(\phi, n) = \mathrm{Tr}[\rho F_{\mathrm{het}}(\phi, n)]$. Such measurements introduce significant extrinsic uncertainty into the measurement result [1]. Thus with standard detection techniques, states with small intrinsic phase uncertainty offer only a modest increase in efficiency over coherent states with the same mean photon number [2].

In this work I am proposing a new technique: adaptive single-shot phase measurements. As I show, such measurements can be much closer to canonical measurements than standard measurements (hence the title of this Letter). The basic idea is to measure the *estimated* phase quadrature of the system by homodyne detection, where the estimate is based on the photocurrent record *so far* from the *single* pulse. That is to say, the local oscillator phase is continuously adjusted by a feedback loop to be in quadrature with the estimated system phase over the course of a single measurement (see Fig. 1). The first part of this Letter explains how this estimate could be made in general. I then present some numerical results for adaptive phase measurements of coherent states. Finally, I present results for a special case which can be solved analytically, in which the adaptive phase measurement is strictly as good as a canonical measurement.

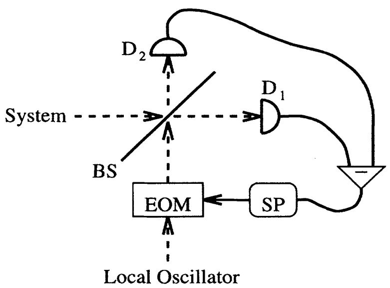

FIG. 1. The adaptive phase measurement scheme. Light beams are indicated by dashed lines and electronics by solid lines. BS denotes a 50/50 beam splitter, D_1 and D_2 photodetectors, and SP a signal processor. The local oscillator phase is controlled by an electro-optic modulator (EOM).

Photodetection with a local oscillator.—An adaptive measurement requires one to estimate the phase of the system based on the measurement record so far. In order to treat this, we require the quantum measurement theory of photodetection with a strong local oscillator, over a finite time interval. This I have derived in generality in Ref. [6], using the recently published theory of linear quantum trajectories of Goetsch and Graham [7]. For simplicity, let the single mode to be measured be prepared initially in a cavity in state ρ. Let the light leak out through an end mirror with decay rate of unity. The emitted light is sent through a 50-50 beam splitter, with a strong local oscillator of complex amplitude $\beta(t) = |\beta|e^{i\Phi(t)}$ entering at the other port. The mean fields at detectors D_2, D_1 are thus proportional to $\beta \pm \langle a\rangle e^{-t/2}$, respectively, where a is the annihilation operator in the cavity mode. The signal photocurrent $I(t)$ in the interval $[t, t+\delta t)$ is defined in terms of the difference between the photocounts $\delta N_2, \delta N_1$ at the two detectors:

$$I(t) = \lim_{|\beta|\to\infty} \frac{\delta N_2(t) - \delta N_1(t)}{|\beta|\delta t}, \quad (5)$$

as in Ref. [5]. It is easy to see that the mean value of this current can be written in terms of $\langle a\rangle = \mathrm{Tr}[a\rho]$ as

$$\langle I(t)\rangle = e^{-t/2}\langle a e^{-i\Phi(t)} + a^\dagger e^{i\Phi(t)}\rangle. \quad (6)$$

It is useful to introduce a new symbol $\mathbf{I}_{[0,t)}$ for the complete photocurrent record $\{I(s) : 0 \le s < t\}$. The quantum measurement theory we require is the POVM for the record $\mathbf{I}_{[0,t)}$ from time 0 (the time of preparation) to time t. This gives the probability for getting the result $\mathbf{I}_{[0,t)}$ given the initial state ρ. Note that $\mathbf{I}_{[0,t)}$ is a continuous infinity of a real number—a very complicated object. Fortunately, it turns out [6] that the POVM depends only on two complex functionals of $\mathbf{I}_{[0,t)}$. These two sufficient statistics are

$$\mathcal{F}_1[\mathbf{I}_{[0,t)}] = \int_0^t e^{i\Phi(s)} e^{-s/2} I(s)\,ds, \quad (7)$$

$$\mathcal{F}_2[\mathbf{I}_{[0,t)}] = -\int_0^t e^{2i\Phi(s)} e^{-s}\,ds. \quad (8)$$

It might be thought that the second functional does not even depend on $\mathbf{I}_{[0,t)}$, but it does if the local oscillator phase $\Phi(t)$ depends on $\mathbf{I}_{[0,t)}$ as in adaptive measurements. Denoting the measured values of these functionals R_t and S_t, respectively, the POVM is [6]

$$F_t(R_t, S_t) = P_0(R_t, S_t) G_t(R_t, S_t), \quad (9)$$

where $P_0(R_t, S_t)$ is a positive function defined later, and $G_t(R_t, S_t)$ is a positive operator given by

$$G_t = \exp(\tfrac{1}{2} S_t a^{\dagger 2} + R_t a^\dagger)\exp(-a^\dagger a t) \times \exp(\tfrac{1}{2} S_t^* a^2 + R_t^* a). \quad (10)$$

The POVM (9) is normalized as usual so that

$$\int d^2R_t d^2S_t P(R_t, S_t) = 1, \quad (11)$$

where $P(R_t, S_t) = \mathrm{Tr}[\rho F_t(R_t, S_t)]$ is the *actual* PDF for obtaining the results R_t, S_t given the initial state ρ. By contrast, the function $P_0(R_t, S_t)$ in Eq. (9) can be thought of as the *ostensible* PDF for R_t and S_t [6]. It is the PDF they would have if $dW(t) = I(t)dt$ were a Wiener process [8] satisfying $dW(t)^2 = dt$. Explicitly,

$$P_0(R_t, S_t) = \int d\mathbf{I}_{[0,t)} P_0(\mathbf{I}_{[0,t)}) \delta^{(2)}(R_t - \mathcal{F}_1[\mathbf{I}_{[0,t)}]) \times \delta^{(2)}(S_t - \mathcal{F}_2[\mathbf{I}_{[0,t)}]). \quad (12)$$

Here $P_0(\mathbf{I}_{[0,t)})$ equals the continuously infinite product of ostensible distributions for each instantaneous current $I(s)$ over each interval $[s, s+ds)$

$$P_0[I(s)] = \sqrt{ds/2\pi}\,\exp[-\tfrac{1}{2} ds I(s)^2]. \quad (13)$$

Recovering the standard result.—The theory presented here applies to any sort of detection with a large local oscillator. This includes heterodyne detection for which the local oscillator phase $\Phi(t)$ cycles rapidly in time at rate $\Delta \gg 1$. In this case S_t does not depend on $\mathbf{I}_{[0,t)}$, and from (8) we find that $S_t \to 0$ as $\Delta \to \infty$. The measurement result is thus R_t, which from (7) and (12) has the *ostensible* statistics of the random variable

$$R_t = \int_0^t e^{-s/2} e^{-i\Delta s} dW(s), \quad (14)$$

where $dW(t)$ is a Wiener increment. Being the (continuous) sum of Gaussian random variables, R_t must be a Gaussian random variable. The rapid phase rotation at rate $\Delta \to \infty$ ensures that is has no preferred phase. Writing $A = R_\infty$, it is easy to show from Eq. (14) that the expected value of $|A|^2$ is 1. These three constraints define the ostensible PDF for the final result A at $t = \infty$:

$$P_0^{\mathrm{het}}(A) = \pi^{-1}\exp(-|A|^2). \quad (15)$$

Substituting this into Eq. (9) and using the fact that $\lim_{t\to\infty} \exp(-a^\dagger at) = |0\rangle\langle 0|$ yields the effect

$$F_\infty^{\rm het}(A) = \pi^{-1} e^{-|A|^2} \exp(Aa^\dagger)\,|0\rangle\langle 0| \exp(A^* a)\,. \quad (16)$$

A little operator algebra confirms that this is identical to the previously stated result (4). The natural phase estimate is thus $\phi = \arg A$. This can be understood as follows. The *actual* mean photocurrent (6) has two counterrotating complex terms. The rotation of the kernel of the integral (7) reinforces that of the second term (which thus averages to zero) but cancels that of the first term, leaving $\langle A\rangle = \int_0^\infty \langle I(t)\rangle e^{i\Phi(t)-t/2}dt = \langle a\rangle$. This result assumes a rapidly varying $\Phi(t)$; in general the second functional (8) is also required to estimate ϕ.

Estimating the phase.—Consider a state ρ_0 with a phase distribution centered around zero. Such a state could be used for communication by encoding a number $\varphi \in [0, 2\pi)$ as a phase shift by the unitary operator $U(\varphi) = \exp(-ia^\dagger a\varphi)$. The PDF for the receiver to get results R_t, S_t for a given φ is thus

$$P(R_t, S_t|\varphi) = \mathrm{Tr}[U(\varphi)\rho_0 U^\dagger(\varphi) F_t(R_t, S_t)]\,. \quad (17)$$

The receiver, who wishes to estimate the phase φ, can use Bayesian statistics [3] to find the posterior PDF

$$P(\varphi|R_t, S_t) = \mathcal{N}^{-1} P(R_t, S_t|\varphi) P_{\rm prior}(\varphi)\,, \quad (18)$$

where $P_{\rm prior}(\varphi)$ expresses the *prior* knowledge the receiver has about φ, and $\mathcal{N}$ is a normalization factor. To be unbiased, we assume that the receiver knows ρ_0 but has no idea about φ, so that the prior PDF $P_{\rm prior}(\varphi)$ is flat [3]. Then it follows from Eq. (9) that at time t the maximum likelihood estimate (MLE) $\hat\varphi_t$ for φ is that $\hat\varphi_t(R_t, S_t)$ which maximizes the likelihood function

$$L_t(\hat\varphi_t) = \mathrm{Tr}[\rho_0 U^\dagger(\hat\varphi_t) G_t(R_t, S_t) U(\hat\varphi_t)]\,. \quad (19)$$

The MLE is the most convenient estimate of phase because, unlike other estimates (such as the mean), it does not suffer from ambiguity due to the cyclic nature of φ.

Adaptive measurements.—The above result for the MLE $\hat\varphi_t$ of the phase is true no matter how the local oscillator phase $\Phi(t)$ varies. Thus we can use this MLE which emerges from the processing of the signal $\mathbf{I}_{[0,t)}$ in a feedback loop to control $\Phi(t)$ as shown in Fig. 1. The suggestion here is that the local oscillator phase be controlled to be in quadrature with the current estimated phase of the system, so that in the next instant of time the apparatus will make a homodyne measurement of the estimated phase quadrature. Explicitly,

$$\Phi(t) = \hat\varphi_t(R_t, S_t) + \pi/2\,. \quad (20)$$

Another way of looking at this is that the receiver attempts to make a *null measurement* of phase. At small t, the receiver has very little information. Thus initially $\hat\varphi_t$, and hence $\Phi(t)$, varies wildly in time. This has the same effect as the rapidly cycling $\Phi(t)$ in heterodyne detection: all quadratures are sampled equally. As more information is acquired the phase estimate improves and approaches the value which is finally used as the result of the measurement, $\phi = \hat\varphi_t(R_\infty, S_\infty)$.

In order to understand this process better, it is helpful to look at an example where the function $L_t(\hat\varphi_t)$ has a relatively simple form. If ρ_0 is the coherent state $|r\rangle\langle r|$ with r real, then one finds that one should maximize

$$\ln L_t(\hat\varphi_t) = \mathrm{Re}[r^2 S_t^* e^{2i\hat\varphi_t} + 2rR_t^* e^{i\hat\varphi_t}] + c\,, \quad (21)$$

where c is a constant (independent of $\hat\varphi_t$). If $r \gg 1$ then for short times $t \lesssim r^{-2}$ the MLE $\hat\varphi_t$ is approximately equal to $\arg R_t$. This is because for short times $|R_t| \sim \sqrt{t}$, while $S_t \ll t$ due to the rapid variation of $\Phi(t)$. This estimate ($\arg R_t$) is the same as that for heterodyne detection, as expected. For longer times the estimated phase settles down, S_t becomes significant, and hence $\hat\varphi_t$ becomes approximately constant at $\arg\sqrt{S_t}$ (with the ambiguity resolved by the phase of R_t). During this stage the measurement is effectively a homodyne measurement of the phase quadrature.

Even with the relatively simple form (21) of $L_t(\hat\varphi_t)$ for coherent states, it is not possible to solve the scheme analytically. Rather, an ensemble of stochastic numerical simulations is needed. The best way to do this is by using the theory of *nonlinear* quantum trajectories [5,9]. In this particular case the system state need not be simulated; it is simply $|re^{i\varphi - t/2}\rangle$. From this, the photocurrent is generated with the correct *actual* statistics by

$$I(t) = e^{-t/2} 2r\cos[\varphi - \Phi(t)] + \xi(t)\,, \quad (22)$$

where $\xi(t)$ is Gaussian white noise [8]. For each simulation, R_t and S_t are calculated and used in Eqs. (20) and (21), and the final MLE $\phi = \hat\varphi_\infty$ stored. An ensemble size of 100 for $r = 50$ gave the mean squared difference between actual and estimated phases to be

$$E[(\phi - \varphi)^2]_{\rm adapt} = (1.0 \pm 0.2) \times 10^{-4}\,. \quad (23)$$

This is half the variance of the standard result of $(2r^2)^{-1}$ [2]. Within statistical error, it is equal to the error of a canonical phase measurement, $(4r^2)^{-1}$ [2]. This is not unexpected, since for $r \gg 1$ the vast majority of the measurement time is spent in an effective homodyne measurement of the phase quadrature. For states with smaller intrinsic phase uncertainty than coherent states the advantage of adaptive measurements over standard measurements would of course be more dramatic.

An analytic example.—There is one case in which the adaptive measurement may be treated analytically: if the system is known to contain at most one photon. This could occur if the cavity mode were excited by a single atom, in which case the phase of the field is equal to the original phase of the dipole of the atom. Since the harmonic oscillator truncated at one photon is equivalent to a two-level atom, the letters TLA will be used to distinguish this case. First I consider canonical and standard measurements. Projecting (2) into the subspace spanned by $\{|0\rangle, |1\rangle\}$ yields the canonical POVM

$$F_{\rm can}^{\rm TLA} = \frac{1}{2\pi}|\phi\rangle\langle\phi|\,, \qquad |\phi\rangle = |0\rangle + e^{i\phi}|1\rangle\,. \quad (24)$$

Similarly, the standard POVM (3) becomes

$$F_{\rm std}^{\rm TLA}(\phi) = \frac{\sqrt{\pi}}{2} F_{\rm can} + \left(1 - \frac{\sqrt{\pi}}{2}\right)\frac{\hat{1}}{2\pi}. \qquad (25)$$

That is to say, the standard technique has an efficiency of $\sqrt{\pi}/2 \simeq 88\%$, in the sense that the same POVM would arise from a canonical measurement that worked 88% of the time, and that gave a random answer on the interval $[0, 2\pi)$ the other 22% of the time.

Because of the isomorphism between the at-most-one-photon field and the two-level atom, it is permissible to replace the annihilation operator a with the lowering operator $|0\rangle\langle 1|$ in all operators. One thus finds the very simple expression for the likelihood function (19):

$$L_t(\hat{\varphi}_t) = \mathrm{Re}[2\langle 1|\rho_0|0\rangle R_t^* e^{i\hat{\varphi}_t}] + c', \qquad (26)$$

where c' is a constant. This does not depend on S_t, so for any ρ_0 which has a mean phase of zero (i.e., for which $\langle 1|\rho_0|0\rangle$ is real and positive), one has simply $\hat{\varphi}_t = \arg R_t$. Thus using Eq. (20) the local oscillator phase is set to be $\Phi(t) = \arg R_t + \pi/2$. The integral equation $R_t = \mathcal{F}_1[\mathbf{I}_{[0,t)}]$ (7), where $\mathbf{I}_{[0,t)}$ has the ostensible distribution (12), is thus equivalent to the following Itô stochastic differential equation:

$$dR_t = e^{-t/2} i(R_t/|R_t|)dW(t), \qquad (27)$$

where as before $dW(t) = I(t)dt$ is ostensibly a Wiener increment. Using the Itô calculus [8] for $|R|^2$ and $\arg R$ quickly yields the solution

$$R_t = \sqrt{1 - e^{-t}} \exp\left[i \int_0^t \frac{e^{-s/2}}{\sqrt{1 - e^{-s}}} dW(s) \right]. \qquad (28)$$

For $t \longrightarrow \infty$ one finds $A = R_\infty$ given by

$$A = \exp(i\phi), \qquad \phi = \int_0^\infty \frac{I(t)dt}{\sqrt{e^t - 1}}. \qquad (29)$$

This ϕ is ostensibly a completely random phase (because the integrand diverges as $t \longrightarrow 0$), so $P_0(\phi) = (2\pi)^{-1}$. Equation (29) shows that the measurement result contains no intensity information, only phase information, as desired. Indeed, substituting these results into Eq. (9) with a replaced by $|0\rangle\langle 1|$ gives the POVM for ϕ

$$F_{\rm adapt}^{\rm TLA}(\phi) = (2\pi)^{-1} \exp(e^{i\phi}|1\rangle\langle 0|)|0\rangle\langle 0| \exp(e^{-i\phi}|0\rangle\langle 1|)$$
$$= F_{\rm can}^{\rm TLA}(\phi). \qquad (30)$$

Thus for the two-level atom (or equivalently for a field with at most one photon), a simple adaptive measurement scheme can produce a canonical measurement of the phase, which is significantly better than the best result with no feedback (25).

The results presented here show that adaptive phase measurements can be close to canonical phase measurements for states with both low and high photon numbers. The key idea is to optimize the measurement at each instant of time by using the MLE of the phase to control the local oscillator phase (20). It should be understood that I have not shown that this is the globally optimal algorithm. Also, the present algorithm assumes the receiver has complete knowledge of the initial state except for its phase. Alternative situations will be considered in future work. The important result from this paper is that by using feedback to create an adaptive phase measurement, a great improvement over standard techniques may be found. If there is at most one excitation in the system, the adaptive technique is as good as a canonical one. Since the feedback requires only electronics and an electro-optic modulator, it should be experimentally feasible, and would represent a fundamental methodological advance over standard phase measurements.

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