Applied Physics 190c: Homework #1

(Dated: April 13, 2017)

Due: Friday, April 21st. Deposit in the INBOX outside of Watson 264.

1. **Reading**

   Section II and Appendix A of Clerk2010 [1]. You might also might want to read some of Chapter 1 in Carmichael [2] and Chapter 3 in Barnett [3].

2. **(60 points) Quantum noise spectral density of a harmonic oscillator in thermal equilibrium**

   In class we considered the quantum noise spectral density (NSD) of hermitian operators (observables) like the position of a harmonic oscillator. For non-hermitian operator $\hat{A}$, we generalize the quantum noise spectral density for a stationary system as,

$$S_{AA}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{i\omega\tau} \langle \hat{A}^{\dagger}(\tau)\hat{A}(0)\rangle. \tag{1}$$

   Here and in what follows we conform to the definition of the Fourier transform operator of a time-dependent operator as found in Clerk2010 [1],

$$\hat{f}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} \hat{f}(t), \tag{2}$$

$$\hat{f}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega). \tag{3}$$

   The operators $\hat{f}(\omega)$ should not be confused with (Schrödinger) continuum operators that might be a function of frequency. We will try to be careful with our notation to avoid such confusion.

   (a) In many cases one will be dealing with a continuum of modes, such as in the radiation modes of a cavity. In such cases it is sometimes more straightforward to consider correlations in the frequency domain. Show that the NSD of operator $\hat{A}(t)$ can be written in the Fourier domain as,

$$S_{AA}(\omega) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \langle \hat{A}^{\dagger}(\omega)\hat{A}(\omega')\rangle. \tag{4}$$

Consider a single-mode optical cavity (annihilation operator $\hat{a}$) weakly coupled to a thermal bath with average photon occupancy $\bar{n}$ near the cavity resonance frequency (ω_c). Assume that the bath can be approximated as a single-mode waveguide with corresponding continuum annihilation operators $\hat{r}_{\omega_r}$.

(b) The Hamiltonian of the coupled cavity-waveguide system is,

$$\hat{H} = \hbar\omega_c \hat{a}^{\dagger}\hat{a} + \int \mathrm{d}\omega_r \hbar\omega_r \hat{r}^{\dagger}_{\omega_r}\hat{r}_{\omega_r} + \left(\int \mathrm{d}\omega_r \hbar\kappa_r(\omega_r)\hat{r}^{\dagger}_{\omega_r}\hat{a} + \text{h.c.}\right). \tag{5}$$

Considering the form of the above Hamiltonian, what is the appropriate commutation relation for the continuum bath operators, $[\hat{r}_{\omega}, \hat{r}^{\dagger}_{\omega'}]$? What are the units of the coupling coefficients, $\kappa_r(\omega_r)$?

(c) Assuming that the coupling coefficients (κ_r) are independent of frequency and the coupling is weak (Markov and Born approximations), show that the corresponding Quantum Langevin Equation (QLE) for the Heisenberg operator $\hat{a}(t)$ is,

$$\dot{\hat{a}}(t) = (-i\omega_c - \kappa/2)\,\hat{a}(t) + \sqrt{\kappa}\hat{a}_{\text{in}}(t), \tag{6}$$

where $\kappa \equiv 2\pi|\kappa_r|^2$, $\hat{a}_{\text{in}}(t) \equiv \int (\mathrm{d}\omega_r/2\pi)\, e^{-i\omega_r t}\left(-ie^{-i\phi_r}\sqrt{2\pi}\hat{r}_{\omega_r}\right)$, $\kappa_r = |\kappa_r|e^{i\phi_r}$, and $\hat{r}_{\omega_r} \equiv \hat{r}_{\omega_r}(t=0)$ are Schrödinger operators. This can be derived from the Heisenberg equations of motion for the cavity and waveguide operators, and corresponds to the classical coupled mode equations for the cavity-waveguide system (see, for instance, Haus [4]) with quantum noise input.

(d) Transform the QLEs for Heisenberg operators $\hat{a}(t)$ and $\hat{a}^{\dagger}(t)$ into the Fourier domain, and solve for $\hat{a}(\omega)$ and $\hat{a}^{\dagger}(\omega)$ in terms of $\hat{a}_{\text{in}}(\omega)$ and $\hat{a}^{\dagger}_{\text{in}}(\omega)$, respectively. Note that these are Fourier operators as defined by eqs. (2).

For a thermal bath, the correlation of the (Schrödinger picture) waveguide operators are given by,

$$\langle \hat{r}^\dagger_\omega \hat{r}_{\omega'} \rangle = \bar{n}\delta(\omega-\omega'), \tag{7}$$

$$\langle \hat{r}_\omega \hat{r}^\dagger_{\omega'} \rangle = (\bar{n}+1)\,\delta(\omega-\omega'), \tag{8}$$

$$\langle \hat{r}^\dagger_\omega \hat{r}^\dagger_{\omega'} \rangle = \langle \hat{r}_\omega \hat{r}_{\omega'} \rangle = 0. \tag{9}$$

This is not easy to show (see Ch. 3.5 of Barnett [3]), but maintains Bose statistics for the continuum of harmonic oscillators in the thermal bath.

(e) What are the corresponding correlation functions between the Fourier operators of the bath noise inputs, $\langle \hat{a}^\dagger_{\text{in}}(\omega)\hat{a}_{\text{in}}(\omega')\rangle$, $\langle \hat{a}_{\text{in}}(\omega)\hat{a}^\dagger_{\text{in}}(\omega')\rangle$, $\langle \hat{a}_{\text{in}}(\omega)\hat{a}_{\text{in}}(\omega')\rangle$, and $\langle \hat{a}^\dagger_{\text{in}}(\omega)\hat{a}^\dagger_{\text{in}}(\omega')\rangle$? Show that the thermal bath noise inputs are thus delta-correlated, $\langle \hat{a}^\dagger_{\text{in}}(t)\hat{a}_{\text{in}}(t')\rangle = \bar{n}\delta(t-t')$ and $\langle \hat{a}_{\text{in}}(t)\hat{a}^\dagger_{\text{in}}(t')\rangle = (\bar{n}+1)\,\delta(t-t')$.

Use these relations to evaluate the steady state NSD of the Heisenberg operator $\hat{a}(t)$. For a relatively high-Q cavity, note whether this two-sided power spectral density has dominantly positive or negative frequency contributions.

(f) Evaluate the steady state NSD of operator $\hat{a}^\dagger(t)$. Note whether this two-sided power spectral density has dominantly positive or negative frequency contributions. In the language of Clerk2010 this represents the ability of the cavity to absorb energy from the bath.

(g) Evaluate the corresponding (steady-state) NSD and autocorrelation of the effective "position" quadrature of the cavity, $\hat{x}_{\lambda=0}(t)$ [note, the general quadrature amplitude is defined as $\hat{x}_\lambda = (\hat{a}e^{-i\lambda} + \hat{a}^\dagger e^{+i\lambda})/\sqrt{2}$]. In the language of Clerk2010, what does this say about the cavity's ability to emit/absorb energy into/from the thermal bath?

(h) **Bonus:** If one were to observe the noise power of the cavity via the coupling waveguide, how would one have to modify the calculated NSD in part (f) to obtain the measured NSD?

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[1] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, arXiv:0810.4729 (2010).

[2] H. J. Carmichael, *Statistical Methods in Quantum Optics 1: Master Equations and Focker-Planck Equations*, Texts and Monographs in Physics (Springer-Verlag, New York, NY, 1999).

[3] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics*, Oxford Series in Optical Imaging and Sciences (Oxford University Press Inc., New York, NY, 1997).

[4] H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, Englewood Cliffs, New Jersey 07632, 1984), 1st ed.