Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation

C. W. Gardiner and M. J. Collett
Physics Department, University of Waikato, Hamilton, New Zealand
(Received 29 October 1984)

We develop a formulation of quantum damping theory in which the explicit nature of inputs from a heat bath, and of outputs into it, is taken into account. Quantum Langevin equations are developed, in which the Langevin forces are the field operators corresponding to the input modes. Time-reversed equations exist in which the Langevin forces are the output modes, and the sign of damping is reversed. Causality and boundary conditions relating inputs to system variables are developed. The concept of "quantum white noise" is formulated, and the formal relationship between quantum Langevin equations and quantum stochastic differential equations (SDE's) is established. In analogy to the classical formulation, there are two kinds of SDE's: the Ito and the Stratonovich forms. Rules are developed for converting from one to the other. These rules depend on the nature of the quantum white noise, which may be squeezed. The SDE's developed are shown to be exactly equivalent to quantum master equations, and rules are developed for computing multitime-ordered correlation functions with use of the appropriate master equation. With use of the causality and boundary conditions, the relationship between correlation functions of the output and those of the system and the input is developed. It is possible to calculate what kind of output statistics result, provided that one knows the input statistics and provided that one can compute the system correlation functions.

I. INTRODUCTION

The treatment of inputs and outputs in quantum systems has a long history. In the context of quantum field theory, the most explicit formulation has been that of Lehman, Symanzik, and Zimmerman[1] (known as LSZ), in which operators for "in" and "out" quantum fields are formulated, and the S matrix derived as the unitary transformation relating the in and out operators. The matrix elements of this S matrix are then the scattering amplitudes, measurable by scattering experiments. The S-matrix elements are related by a "reduction formula" to the time-ordered products of vacuum expectation values of field operators. There is a range of approximate methods of tackling the computation of S-matrix elements, with differing degrees of simplicity and reliability.

In the theory of quantum amplification, we are most interested in how a well-defined subsystem, the amplifier, relates the output to the input. Caves[2] took the S-matrix point of view, and defined a theory of linear amplifiers, in which the S matrix related the output fields linearly to the input fields, and thus, without having to consider any of the details of the amplifying system, was able to derive fundamental limits on amplification for such linear amplifiers.

As far as quantum optics is concerned, these points of view are rather far from that normally used. The amplifying device itself can usually be described quite well, but the computation of inputs and outputs is rather poorly formulated. One normally thinks of the system as being in contact with a heat bath, which most usually consists of the vacuum modes of the electromagnetic field. By eliminating these modes, one obtains a damped equation of motion for the system, to which are added quantum noise terms. If required an additional classical deriving field is usually added separately, though it is clearly the result of one or more of the modes of the electromagnetic field being in a coherent state, rather than the vacuum. The formal equivalence of a bath mode being in a coherent state, and the addition of a classical driving field, is both rigorous and well known, and so in this case this procedure is merely *aesthetically* unsatisfactory in its separation of two different aspects of the electromagnetic field. However, no procedure has been developed so far which can cope adequately with the possibility of input fields which are neither coherent nor thermal, for example, squeezed fields, which are now of increasing interest.

The output modes are calculated usually by particular methods; e.g., in resonance fluorescence the radiated electromagnetic field is calculated from the solutions of Maxwell's equations,[3,4] while in the case of cavities linear propagation equations can be used to relate in operators to out operators.[5] Yurke and Denker[6] in their formulation of quantum circuit theory have demonstrated the intimate connection between the noise in conventional damped Heisenberg equations of motion, and the *input* operators for an infinite transmission line connected to a resonant circuit. As well as this, they showed that the boundary condition between the infinite transmission line and the finite resonant circuit gives rise to a time-reversal phenomenon, in which the damping changes sign, and the system is driven by the output operators.

This paper is devoted to the formulation of an idealized "quantum white-noise" method of looking at quantum systems. In Sec. II we formulate our system in terms of a somewhat idealized class of Hamiltonians, in which a finite "system" is coupled to a "heat bath" of harmonic oscillators derived by the following three assumptions.

(i) A particular class of system-bath interactions, which are linear in the bath operators.

(ii) The rotating-wave approximation is made.

(iii) The bath spectrum is assumed flat, and the coupling constant independent of frequency.

These are assumptions which are almost universally made in quantum optics, but have less validity at low frequencies, which will be treated in the second paper of this series.

Using these assumptions it is possible to derive in and out operators, which are expressed in terms of the bath operators evaluated at the remote past and future, respectively, and to derive quantum Langevin equations by a rather standard method in which the noise term is expressed in terms of the in operators. The out operators can be used in the same way to produce the "time-reversed" quantum Langevin equation, in which the damping is of opposite sign, and the out operator now appears as the noise term.

In Sec. III we formulate precisely the concept of quantum white noise, which is an approximation to the usual thermal state of an optical field. We then define quantum stochastic integration in terms of "quantum Wiener process" corresponding to quantum white noise. As in classical stochastics, the singular nature of quantum white noise makes it mandatory to define the precise kind of integration chosen (i.e., either Ito or Stratonovich); corresponding to these different kinds of integrations, one finds different definitions of "quantum stochastic differential equations" (QSDE's), either Ito or Stratonovich.

In the Ito form, the white-noise increment is both uncorrelated with and commutes with the system operators; in the Stratonovich form this is not so, although both commutators and correlations can be specified.

The quantum Langevin equations derived in Sec. II are valid for any kind of statistics of the input field, however in practice the kind of input most commonly found is thermal (to which may be added a coherent part). A thermal field has a Planck spectrum, and is therefore not quantum white noise—any replacement of such an input field by quantum white noise must involve some kind of approximation.

We give two ways of replacing the input field by quantum white noise whose justification is their correspondence to the master equation, a correspondence which is demonstrated in Sec. IV. In one, the input field is simply replaced by a quantum-white-noise source, in the other, a separate noise source is chosen for each transition possible in the system.

The QSDE's are written down as Ito equations, since this is the only kind that can be defined precisely, and the properties of Stratonovich QSDE's are only able to be demonstrated by means of the Ito form.

In Sec. IV we show how to derive from QSDE's (in the Ito form) the appropriate master equations. The equivalence is exact: approximations usually necessary to derive master equations have already been made either in the derivation of the Langevin equation or in the replacement of the input field by quantum white noise. The simple master equation and Lax's[7] master equation both arise out of the corresponding QSDE.

To complete the master-equation description, we also show how to compute time correlation functions by the standard formulas, as given by several authors. These are exact consequences of the QSDE. It strikes the authors as somewhat curious, though, that not every correlation function can be computed, but only those expressible as the mean of a product of an anti-time-ordered product followed by a time-ordered. These appear to be the only *measurable* quantities, or at least, the only quantities expressible in terms of repeated measurements on the system, and indeed are the only quantities which can arise from output measurements. But mathematically, the QSDE seems to provide (in principle) a way of computing correlation functions of arbitrary time ordering, which the master equation does not.

Section V is devoted to the computation of the relation between the input, output, and internal correlation functions. We show that arbitrary correlation functions of the output fields can be computed in terms of the time-ordered correlation functions of the system, which can in turn be computed using the methods of Sec. IV. These results are relatively new; they are implied in the work of Kimble *et al.*[3] on resonance fluorescence, but not in the generality presented here.

We summarize in Sec. VI the relationship between this work and previous work, and indicate the problems to be solved in future papers.

II. QUANTUM LANGEVIN EQUATIONS

A. Background

In this section we shall be considering the rather conventional picture of a system interacting with a heat bath, in the form

$$\begin{aligned} H &= H_{\text{sys}} + H_B + H_{\text{int}} \ , \\ H_B &= \hbar \int_{-\infty}^{\infty} d\omega\, \omega b^\dagger(\omega) b(\omega) \ , \\ H_{\text{int}} &= i\hbar \int_{-\infty}^{\infty} d\omega\, \kappa(\omega)[b^\dagger(\omega)c - c^\dagger b(\omega)] \ , \end{aligned} \tag{2.1}$$

where the $b(\omega)$ are boson annihilation operators for the bath, with

$$[b(\omega), b^\dagger(\omega)] = \delta(\omega - \omega') \tag{2.2}$$

and c is one of several possible system operators. We do not specify either H_{sys} or the kind of system operators or their commutation relations.

The Hamiltonian (2.1) is, of course an idealization. In practice the range of ω is $(0,\infty)$, but a range of $(-\Omega,\infty)$ can arise when we go into a frame rotating with angular frequency Ω as is common in quantum optics, and it is certainly the case that Ω is very large compared with the typical bandwidths obtained. The coupling of the bath to the system is through H_{int}, which is linear in $b(\omega)$ and $b^\dagger(\omega)$. This is a common assumption, but is not the most general possibility. We have not investigated the consequences of a more general coupling, which we leave for a later work. This linear coupling is, however, a very important factor in our development. For simplicity we have considered only one bath—we can easily relax this condition, however.

Historically, this model of quantum damping by coupling to a bath is very well developed, and methods of deriving damping equations have been developed by many authors. Usual methods rely on deriving a master equation for the system density matrix, and bypass the concept of a Langevin equation[7–12] and the derivation of the master equation requires a perturbative limit. The rigorous formulation of this kind of approach is that of quantum-dynamical semigroups[13,14] and retains, in a rigorous asymptotic form, this essentially perturbative character. The Langevin method which we shall use was first introduced by Haken[15] who has used it extensively in linear problems.

B. Derivation of the Langevin equations

We follow the standard procedure. From (2.1) we derive the Heisenberg equations of motion for $b(\omega)$, and an arbitrary system operator a. They are

$$\dot{b}(\omega)=-i\omega b(\omega)+\kappa(\omega)c\ , \tag{2.3}$$

$$\dot{a}=-\frac{i}{\hbar}[a,H_{\text{sys}}]+\int d\omega\,\kappa(\omega)\{b^\dagger(\omega)[a,c]-[a,c^\dagger]b(\omega)\} \tag{2.4}$$

and we solve (2.3) to obtain

$$b(\omega)=e^{-i\omega(t-t_0)}b_0(\omega)+\kappa(\omega)\int_{t_0}^{t}e^{-i\omega(t-t')}c(t')dt'\ . \tag{2.5}$$

Here $b_0(\omega)$ is the value of $b(\omega)$ at $t=t_0$; it is some kind of initial value, and has the same commutation relations as $b(\omega)$. We substitute in (2.4) to obtain

$$\begin{aligned}\dot{a}=-\frac{i}{\hbar}[a,H_{\text{sys}}]&+\int d\omega\,\kappa(\omega)\{e^{i\omega(t-t_0)}b^\dagger(\omega)[a,c]\\&-[a,c^\dagger]e^{-i\omega(t-t_0)}b_0(\omega)\}\\&+\int d\omega[\kappa(\omega)]^2\int_{t_0}^{t}dt'\{e^{i\omega(t-t')}c^\dagger(t')[a,c]\\&-[a,c^\dagger]e^{-i\omega(t-t')}c(t')\}\ .\end{aligned} \tag{2.6}$$

For notational convenience in (2.6) we omit the time argument on the system operators when it is t but write it explicitly otherwise. [Thus $a\equiv a(t)$.]

The equations are exact so far. We now introduce what we shall call the first Markov approximation, that the coupling constant is independent of frequency.

First Markov approximation:

$$\kappa(\omega)=\sqrt{\gamma/2\pi}\ . \tag{2.7}$$

This approximation can be used to put the Eq. (6) into the form of a damping equation. We use the properties

$$\int_{-\infty}^{\infty}d\omega\,e^{-i\omega(t-t')}=2\pi\delta(t-t') \tag{2.8}$$

and

$$\int_{t_0}^{t}c(t')\delta(t-t')dt'=\tfrac{1}{2}c(t)\ . \tag{2.9}$$

The second result will always hold when (2.8) is achieved as the limit of an integral over a function going smoothly to zero at $\pm\infty$, which is essentially what we are doing here.

We also *define* an in field by

$$b_{\text{in}}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega\,e^{-i\omega(t-t_0)}b_0(\omega) \tag{2.10}$$

which satisfies the commutation relation

$$[b_{\text{in}}(t),b_{\text{in}}^\dagger(t')]=\delta(t-t')\ . \tag{2.11}$$

Using (2.8)–(2.10) we readily derive the quantum Langevin equation

$$\begin{aligned}\dot{a}=-\frac{1}{\hbar}[a,H_{\text{sys}}]-&\left[[a,c^\dagger]\left[\frac{\gamma}{2}c+\sqrt{\gamma}b_{\text{in}}(t)\right]\right.\\&\left.-\left[\frac{\gamma}{2}c^\dagger+\sqrt{\gamma}b_{\text{in}}^\dagger(t)\right][a,c]\right]\ .\end{aligned} \tag{2.12}$$

Although this kind of equation and derivation has been used for many years on linear systems, it has not, to our knowledge, ever been taken seriously as a general equation. The equation in the nonlinear case is too intractable to handle directly, but does contain a great deal of information. Some points to notice are the following.

(i) *Damping is included*: the terms proportional to $\frac{1}{2}\gamma c$ and $\frac{1}{2}\gamma c^\dagger$ are in practice damping terms, and arise without any particular specification of the thermal state of the reservoir. The simple example of $a,a^\dagger$ being harmonic-oscillator operators, with $c=a$, gives

$$\dot{a}=-i\omega_0 a-\frac{\gamma}{2}a-\sqrt{\gamma}b_{\text{in}}(t)\ , \tag{2.13}$$

exhibiting this clearly.

We see that the damping is Markovian, i.e., the damping term depends only on the system operators evaluated at time t, not at a previous time, and this arises from the *first Markov approximation*.

(ii) *Noise terms*: the terms depending on $b_{\text{in}}(t),b_{\text{in}}^\dagger(t)$ are to be taken as noise terms. The definition of $b_{\text{in}}(t)$ in terms of the values $b_0(\omega)$ of $b(\omega)$ at time $t=t_0$ ensures that these operators may be freely specified on the same basis as initial conditions. Similarly, we may freely specify the state of the system to be such that, at $t=t_0$, the system and both density operators factorize. It is *not* necessary (nor indeed is it possible) to make any further "independence" assumption. In fact, it is not necessary to make any particular assumption about the initial state of the system at all for the Langevin equation to be valid in the form (2.12). However, the terms $b_{\text{in}}(t),b_{\text{in}}^\dagger(t)$ can only be reasonably interpreted as noise when the state of the system is initially factorized and the state of $b_{\text{in}}(t)$ is incoherent (e.g., a thermal state). In the case that, for example, $b_{\text{in}}(t)$ is in a coherent state, we would have a classical driving field being applied to the system, and other intermediate situations can easily be envisaged. As noted in (i), the existence and form of the damping terms has nothing to do with the state of the bath—damping will occur even with a coherent input.

(iii) *There are alternative forms*: depending on the fact that $\frac{1}{2}\gamma c(t)+\sqrt{\gamma}b_{\text{in}}(t)$ and $\frac{1}{2}\gamma c^{\dagger}(t)+\sqrt{\gamma}b_{\text{in}}^{\dagger}(t)$ commute with all system operators. For, from (2.5), (2.7), and (2.8)

$$\int d\omega\, b(\omega)=b_{\text{in}}(t)+\frac{\sqrt{\gamma}}{2}c(t)\ . \tag{2.14}$$

Since the $b(\omega)$ are bath operators, and commute with all system operators at the same time, this proves the result.

(iv) *Consistency with calculus*: the basic rule of ordinary calculus for noncommuting operators is that of the product—if a_1 and a_2 are two operators

$$\frac{d}{dt}(a_1a_2)=\dot{a}_1a_2+a_1\dot{a}_2\ . \tag{2.15}$$

This rule is also valid for any commutator, i.e., for any A:

$$[A,a_1a_2]=[A,a_1]a_2+a_1[A,a_2]\ . \tag{2.16}$$

Only the first term on the rhs of (2.12) is in the explicit form of a commutator, but the parts involving γ are of the form of commutators multiplied by terms like (2.14), which commute with all system operators. Thus the time derivative of any product will be correctly given by (2.14) and by the substitution $a\rightarrow a_1a_2$ in (2.12). We find that there are no Ito-like terms, even if the input is noise. We come back to this in Sec. III C.

(v) *Out fields*: If we consider $t_1>t$, we can write, analogously to (2.5),

$$b(\omega)=e^{-i\omega(t-t_1)}b_1(\omega)-\kappa(\omega)\int_t^{t_1}e^{-i\omega(t-t')}c(t') \tag{2.17}$$

and similarly define

$$b_{\text{out}}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega(t-t')}b_1(\omega)\ . \tag{2.18}$$

Carrying out the same procedure, we arrive at the *time-reversed* Langevin equation

$$\dot{a}=-\frac{i}{\hbar}[a,H_{\text{sys}}]-\left[[a,c^{\dagger}]\left[-\frac{\gamma}{2}c+\sqrt{\gamma}b_{\text{out}}(t)\right]-\left[-\frac{\gamma}{2}c^{\dagger}+\sqrt{\gamma}b_{\text{out}}(t)\right][a,c]\right] \tag{2.19}$$

in which we see that

$$b_{\text{in}}(t)\rightarrow b_{\text{out}}(t)\ ,\quad \sqrt{\gamma}\rightarrow\sqrt{\gamma}\ ,\quad \frac{\gamma c}{2}\rightarrow-\frac{\gamma c}{2}\ , \tag{2.20}$$

and furthermore

$$\int d\omega\, b(\omega)=b_{\text{out}}(t)-\frac{\sqrt{\gamma}}{2}c(t) \tag{2.21}$$

is derived similarly to (2.13). From this follows the identity that

$$b_{\text{out}}(t)-b_{\text{in}}(t)=\sqrt{\gamma}c(t) \tag{2.22}$$

which also can be used to transform between the forward Langevin equation (2.12) and the time-reversed Langevin equation (2.19).

C. Inputs and outputs, and causality

The quantities $b_{\text{in}}(t)$ and $b_{\text{out}}(t)$ will be interpreted as inputs and outputs to the system. The condition (2.22) can be viewed as a boundary condition, relating input, output, and internal modes, and is the analog of the boundary conditions in propagation equations in the work of Yurke and Denker.[6]

If Eq. (2.12) is solved to give values of the system operators in terms of their past values and those of $b_{\text{in}}(t)$, then it is clear that $a(t)$ is independent of $b_{\text{in}}(t')$ for $t'>t$; that is, the system variables do not depend on the values of the input in the future. Hence we deduce

$$[a(t),b_{\text{in}}(t')]=0,\quad t'>t \tag{2.23}$$

and using similar reasoning, we deduce from Eq. (2.18)

$$[a(t),b_{\text{out}}(t')]=0,\quad t'<t\ . \tag{2.24}$$

Defining the step function $u(t)$ as

$$u(t)=\begin{cases}1, & t>0\\ \frac{1}{2}, & t=0\\ 0, & t<0\end{cases} \tag{2.25}$$

and using Eq. (2.22) we obtain the quite specific results

$$\begin{aligned}[a(t),b_{\text{in}}(t')]&=-u(t-t')\sqrt{\gamma}[a(t),c(t')]\ ,\\ [a(t),b_{\text{out}}(t')]&=u(t'-t)\sqrt{\gamma}[a(t),c(t')]\ .\end{aligned} \tag{2.26}$$

We now have, in principle, a complete specification of a system with input and output. We *specify* the input $b_{\text{in}}(t)$, and solve (2.12) for $a(t)$. We then compute the output from the known $a(t)$ and $b_{\text{in}}(t)$ by use of the boundary condition, Eq. (2.22).

The commutators (2.26) are an expression of quantum causality—that only the future motion of the system is affected by the present input, and that only the future value of the output is affected by the present values of the system operators.

D. Summary

The results of this section, while relevant to the study of a system being driven by a noisy input from a heat bath, are not genuinely stochastic results, since no assumptions have been made concerning the density operator of the bath. In a certain sense, this formalism is a kind of scattering theory, which enables the output fields to be determined from the input fields via the indirect route of their interaction with a system.

III. QUANTUM STOCHASTIC PROCESSES

A. The quantum Wiener process

The fields $b_{\rm in}(t)$ defined in Sec. II will provide the input to the system described by $H_{\rm sys}$. The particular quantum state or ensemble of quantum states of the in operators determines the nature of the input. There will always be, as is well known, some quantum noise arising from the zero-point fluctuations of the input, and depending on the input ensemble, there may be additional noise, such as thermal noise.

The input ensemble which corresponds most closely to a classical-white-noise input is not a thermal ensemble, but one in which the input density operator $\rho_{\rm in}$ is such that

$$\begin{aligned}\mathrm{Tr}[\rho_{\rm in} b_{\rm in}^\dagger(t) b_{\rm in}(t')] &\equiv \langle b_{\rm in}^\dagger(t) b_{\rm in}(t')\rangle \\ &= \bar{N}\delta(t-t')\ , \\ \mathrm{Tr}[\rho_{\rm in} b_{\rm in}(t) b_{\rm in}^\dagger(t')] &\equiv \langle b_{\rm in}(t) b_{\rm in}^\dagger(t')\rangle \\ &= (\bar{N}+1)\delta(t-t')\ ,\end{aligned} \tag{3.1}$$

which corresponds to

$$\begin{aligned}\langle b_0^\dagger(\omega) b_0(\omega')\rangle &= \bar{N}\delta(\omega-\omega')\ , \\ \langle b_0(\omega) b_0^\dagger(\omega')\rangle &= (\bar{N}+1)\delta(\omega-\omega')\ .\end{aligned} \tag{3.2}$$

This corresponds to an ensemble in which the number of quanta per unit bandwidth is constant, and this is not the case in a thermal ensemble, in which in (3.2) $\bar{N}$ would be replaced by $\bar{N}(\omega)$, given by

$$\bar{N}(\omega) = 1/[\exp(\hbar\omega/kT) - 1]\ . \tag{3.3}$$

Thus, to an even larger extent than in classical stochastics, quantum white noise is an idealization, not actually attained in any real system.

To define quantum stochastic integration, we define the quantum Wiener process by

$$B(t,t_0) = \int_{t_0}^t b_{\rm in}(t')dt' \tag{3.4}$$

in which we find that

$$\langle B^\dagger(t,t_0)B(t,t_0)\rangle = \bar{N}(t-t_0)\ , \tag{3.5a}$$

$$\langle B(t,t_0)B^\dagger(t,t_0)\rangle = (\bar{N}+1)(t-t_0)\ , \tag{3.5b}$$

$$[B(t,t_0), B^\dagger(t,t_0)] = t-t_0\ . \tag{3.5c}$$

In addition, we specify that the distribution of $B(t,t_0), B^\dagger(t,t_0)$ is "quantum Gaussian," by which we mean that the density operator is

$$\rho(t,t_0) = (1-e^{-\kappa})\exp\left[-\frac{\kappa B^\dagger(t,t_0)B(t,t_0)}{t-t_0}\right] \tag{3.6}$$

in which

$$\bar{N} = 1/(e^{\kappa}-1)\ . \tag{3.7}$$

It is clear that any normal-ordered moment of order n in $B(t,t_0)$ and $B^\dagger(t,t_0)$ will be a constant times $(t-t_0)^{n/2}$.

The moments of order n with any ordering of $B(t,t_0), B^\dagger(t,t_0)$ will, as a consequence of the commutation relation (3.5c), always be proportional to $(t-t_0)^{n/2}$—a factor of importance in manipulating stochastic differentials.

B. Quantum stochastic integration

In ordinary stochastic integration, there is a choice of the Ito or the Stratonovich definition. The Ito form has some mathematical advantages, which arise from the increment being independent of the integration variable. However, the rules of calculus are not those of ordinary calculus: only in the Stratonovich form is this the case.

In the quantum situation we have the added complication that variables do not commute, which has been treated in Sec. II B 4. We can define both Ito and Stratonovich quantum stochastic integration, and can show that only the Stratonovich form preserves the rules of (noncommuting) calculus. But this can only be proved via a route involving the quantum Ito calculus, so it will be necessary to define both kinds of integration even to use the apparently simpler Stratonovich form.

We therefore define the *quantum Ito integral* by

$$\mathrm{I}\int_{t_0}^t g(t')dB(t') = \lim_{n\to\infty} \sum g(t_i)[B(t_{i+1},t_0) - B(t_i,t_0)]\ , \tag{3.8}$$

where $t_0 < t_1 < t_2 < \cdots < t_n = t$, and the limit is a mean-square limit in terms of the density matrix (3.6). A similar definition can be used for $\int_{t_0}^t g(t')dB^\dagger(t')$.

We assume the Ito increments $dB(t)$ and $dB^\dagger(t)$ commute with $g(t)$, which follows from (2.23) and the definition (3.8), since

$$\begin{aligned}&[g(t_i), B(t_{i+1},t_0) - B(t_i,t_0)] \\ &\quad = -\sqrt{\gamma}\int_{t_i}^{t_{i+1}} dt'\, u(t_i - t')[g(t_i), c(t')] = 0\ .\end{aligned} \tag{3.9}$$

Hence

$$\mathrm{I}\int_{t_0}^t g(t')dB(t') = \mathrm{I}\int_{t_0}^t dB(t')g(t') \tag{3.10}$$

and similarly for integrals with respect to $dB^\dagger(t)$.

If $\bar{N}$ is defined as in Eq. (3.5), we define an Ito QSDE in the form

$$\begin{aligned}da = &-\frac{i}{\hbar}[a, H_{\rm sys}]dt + \frac{\gamma}{2}(\bar{N}+1)(2c^\dagger a c - a c^\dagger c - c^\dagger c a)dt \\ &+ \frac{\gamma}{2}\bar{N}(2cac^\dagger - acc^\dagger - cc^\dagger a)dt \\ &- \sqrt{\gamma}[a, c^\dagger]dB(t) + \sqrt{\gamma}[a, c]dB^\dagger(t)\ .\end{aligned} \tag{3.11}$$

We will show this equation is equivalent to the quantum Langevin equation (2.12); but first, show that the second order calculus rule appropriate to Ito integration,

$$d(ab) = a\ db + a\ db + da\ db\ , \tag{3.12}$$

is true. To do this we need the identities

$$[dB(t)]^2=[dB^\dagger(t)]^2=0\ ,$$
$$dB(t)dB^\dagger(t)=(\bar N+1)dt\ , \tag{3.13}$$
$$dB^\dagger(t)dB(t)=\bar N\,dt\ .$$

All other products, including $dt\,dB(t)$, $dt\,dB^\dagger(t)$, and higher orders are set equal to zero. These will now be derived here, but are easy to derive in exactly the same way as in the nonquantum case.[16]

The proof is then straightforward. We compute da and db using Eq. (3.11), substitute these into into Eq. (3.12) using (3.13), and the result is that (after some rearrangement) $d(ab)$ as derived from the substitution $a\to ab$ in (3.11) is the same as that from (3.12). From Eq. (3.12) we can derive the rules of calculus for any polynomial. In the case that z is a variable which commutes with dz, we obtain

$$df(z)=f'(z)dz+\tfrac{1}{2}f''(z)dz^2 \tag{3.14}$$

which leads to Ito rules. Where z does not commute with dz, Eq. (3.14) can be interpreted in the sense that all products in a power-series expansion of $f'(z)$ and $f''(z)$ are completely symmetrized in terms of z and dz.

C. The quantum Stratonovich integral

The quantum Stratonovich integral is defined by

$$\mathrm{S}\int_{t_0}^{t} g(t')dB(t') = \lim_{n\to\infty}\sum g(\tfrac{1}{2}(t_i+t_{i+1}))[B(t_{i+1},t_0)-B(t_i,t_0)]\ . \tag{3.15}$$

We notice that the Stratonovich increment does *not* commute with $g(t)$, and in fact, using (2.26), it is straightforward to show that we must take

$$\mathrm{S}\int_{t_0}^{t} g(t')dB(t')-\mathrm{S}\int_{t_0}^{t} dB(t')g(t') = \frac{\sqrt{\gamma}}{2}\int_{t_0}^{t} dt'[g(t'),c(t')]\ . \tag{3.16}$$

We can show this more rigorously by deriving the connection between the two kinds of stochastic integral. Let us assume that all operators obey the quantum Ito equation (3.11). Let us define $\bar t_i=\tfrac{1}{2}(t_i+t_{i+1})$, so that we can rewrite (3.15) as

$$\mathrm{S}\int_{t_0}^{t} g(t')dB(t')=\lim_{n\to\infty}\Big[\sum g(\bar t_i)[B(t_{i+1})-B(\bar t_i)] + \sum g(\bar t_i)[B(\bar t_i)-B(t_i)]\Big]\ . \tag{3.17}$$

We then write

$$g(\bar t_i)=g(t_i)+dg(t_i)\ , \tag{3.18}$$

where $dg(t_i)$ is obtained from (3.11), with

$$dt_i=\bar t_i-t_i\ ,$$
$$dB(t_i)=B(\bar t_i)-B(t_i)\ , \tag{3.19}$$

which will be valid to lowest order. We then find that

$$\begin{aligned}\mathrm{S}\int_{t_0}^{t} g(t')dB(t')=\lim_{n\to\infty}\Big[&\sum g(\bar t_i)[B(t_{i+1})-B(t_i)]+\sum g(t_i)[B(\bar t_i)-B(t_i)]\\ &+\sqrt{\gamma}\sum[g(t_i),c^\dagger(t_i)][B(\bar t_i)-B(t_i)][B(\bar t_i)-B(t_i)]\\ &+\sqrt{\gamma}\sum[g(t_i),c(t_i)][B^\dagger(\bar t_i)-B^\dagger(t_i)][B(\bar t_i)-B(t_i)]\Big]\ .\end{aligned} \tag{3.20}$$

We now use (3.13) in the last part, and combine the first two terms into the *Ito* integral, to get

$$\mathrm{S}\int_{t_0}^{t} g(t')dB(t')=\mathrm{I}\int_{t_0}^{t} g(t')dB(t') +\tfrac{1}{2}\sqrt{\gamma}\bar N\int_{t_0}^{t}[g(t'),c(t')]dt' \tag{3.21a}$$

and similarly

$$\mathrm{S}\int_{t_0}^{t} dB(t')g(t')=\mathrm{I}\int_{t_0}^{t} g(t')dB(t') +\tfrac{1}{2}\sqrt{\gamma}(\bar N+1)\int_{t_0}^{t}[g(t'),c(t')]dt'\ , \tag{3.21b}$$

$$\mathrm{S}\int_{t_0}^{t} g(t')dB^\dagger(t')=\mathrm{I}\int_{t_0}^{t} g(t')dB^\dagger(t') -\tfrac{1}{2}\sqrt{\gamma}(\bar N+1)\int_{t_0}^{t}[g(t'),c^\dagger(t')]dt'\ , \tag{3.21c}$$

$$\mathrm{S}\int_{t_0}^{t} dB^\dagger(t')g(t')=\mathrm{I}\int_{t_0}^{t} g(t')dB^\dagger(t') -\tfrac{1}{2}\sqrt{\gamma}\bar N\int_{t_0}^{t}[g(t'),c^\dagger(t')]dt'\ . \tag{3.21d}$$

Substituting for the Ito integral implicit in (3.11), we find the equivalent quantum Stratonovich equations

$$\begin{aligned}(\mathrm{S})da=&-\frac{i}{\hbar}[a,H_{\rm sys}]dt-\frac{\gamma}{2}([a,c^\dagger]c-c^\dagger[a,c])dt\\ &-\sqrt{\gamma}[a,c^\dagger]dB(t)+\sqrt{\gamma}dB^\dagger(t)[a,c]\end{aligned} \tag{3.22}$$

which is exactly of the same form as the *quantum Langevin equation,* Eq. (2.12). We note also that the commutation relation of Sec. II B 3 follows from Eqs. (3.21). Finally, we can compute (S)$d(ab)$ from the corresponding Ito form, and readily verify that

$$(\mathrm{S})d(ab)=a\,db+da\,b \tag{3.23}$$

so that ordinary (noncommuting) calculus is valid, a fact that is very difficult to demonstrate directly from the Stratonovich form.

D. Comparison of the two forms of QSDE

1. Stratonovich

(i) The Stratonovich form is the "natural" physical choice, since it is what arises directly from the physical considerations in Sec. II.

(ii) However, we note that the increment neither commutes with system operators, nor is it stochastically independent of them.

(iii) A direct proof that ordinary calculus is true is difficult to present.

(iv) Because the commutator of the increment and a system variable depends on the precise form of the QSDE, it is not possible to define the quantum Stratonovich integrals without a knowledge of the QSDE.

(v) The QSDE in Stratonovich form should also be valid for nonwhite noise and is the same for any $\bar{N}$ if the noise is white. The only assumption necessary to obtain the Stratonovich QSDE is that of a constant $\kappa(\omega)$, Eq. (2.7).

2. Ito

(i) Not a natural physical choice.

(ii) Increment commutes with and is statistically independent of system operators at the same time.

(iii) Ordinary calculus is not true, but the appropriate Ito calculus is easy to derive.

(iv) Because $\bar{N}$ appears in the QSDE, it is not possible to define the QSDE without knowledge of $\bar{N}$ and bath statistics, and the Ito QSDE (3.11) is exact only from quantum white noise.

E. The situation in which there are several frequencies

Lax, in his development of quantum noise,[7] showed that in an atomic system characterized by several transition frequencies, it is necessary to consider a separate noise source for every frequency. As we have so far considered it, we have only included one noise source, with $\bar{N}$ corresponding to the rotating frame frequency Ω.

The precise correspondence between quantum white noise and the $b_{\mathrm{in}}(t), b_{\mathrm{in}}^{\dagger}(t)$ is a physical question, and the physical formulation has been carried out by Lax, and has been well tested. Lax's master equations and Langevin equations correspond to the following procedure. (We prove the equivalence in Sec. IV.)

We first define eigenstates of the systematic Hamiltonian H_{sys} by

$$H_{\mathrm{sys}}\,|\,i\,\rangle=\hbar\omega_i\,|\,i\,\rangle \tag{3.24}$$

(in which it is understood that $\omega_i>\omega_j$ if $i>j$) and operators A_{ij} by

$$A_{ij}(t_0)=|\,i\,\rangle\langle j\,| \;. \tag{3.25}$$

We then note that we may expand

$$c(t)=\sum_{\substack{l,m\\ l<m}} c_{lm}A_{lm}(t) \;. \tag{3.26}$$

We restrict the summation to $l<m$, since in this case A_{lm} is a lowering operator ($\omega_l<\omega_m$), and in making the rotating-wave approximation, we must multiply $b_{\mathrm{in}}^{\dagger}(t)$ only by lowering operators.

Lax's formulation corresponds to the ansatz

$$c(t)b_{\mathrm{in}}^{\dagger}(t)dt\rightarrow\sum_{\substack{l,m\\ l<m}} c_{lm}A_{lm}(t)dB^{\dagger}(t,\omega_{ml}) \;, \tag{3.27}$$

$$c^{\dagger}(t)b_{\mathrm{in}}(t)dt\rightarrow\sum_{\substack{l,m\\ l<m}} c_{lm}^{*}A_{ml}(t)dB(t,\omega_{ml}) \;, \tag{3.28}$$

where $\omega_{ml}=\omega_m-\omega_l$ and by $dB(t,\omega_{lm})$ we mean a quantum white noise whose $\bar{N}$ is given by

$$\bar{N}=\bar{N}(\omega_{lm}) \tag{3.29}$$

and where $dB(t,\omega_{lm}), dB(t,\omega_{pq})$ are independent of each other if $\omega_{lm}\neq\omega_{pq}$ and are identical if $\omega_{lm}=\omega_{pq}$.

In this case the quantum Ito equation is easily derived to be

$$da=-\frac{i}{\hbar}[a,H_{\mathrm{sys}}]dt-\frac{\gamma}{2}([a,c^{\dagger}]c-c^{\dagger}[a,c])dt-\tfrac{1}{2}\sum_{\substack{l,m\\ l<m}}\bar{N}(\omega_{ml})\,|\,c_{lm}\,|^{2}([[a,A_{lm}],A_{ml}]+[[a,A_{ml}],A_{lm}])dt$$
$$+\sqrt{\gamma}\sum_{\substack{l,m\\ l<m}}\{[a,A_{lm}]c_{lm}dB^{\dagger}(t,\omega_{ml})-[a,A_{ml}]c_{lm}^{*}dB(t,\omega_{ml})\} \;. \tag{3.30}$$

(Here, along with Lax, we have assumed that $\omega_{lm}=\omega_{pq}\Longrightarrow l=p$ and $m=q$. This assumption can be relaxed if necessary.) The assumption that $dB^{\dagger}(t,\omega_{ml})$ and $dB(t,\omega_{pq})$ are independent is not strictly correct, however, it is clear that any correlation will be multiplied by a factor $\exp[i(\omega_{ml}-\omega_{pq})t]$, because the two noise sources are in fact centered at the frequencies ω_{ml} and ω_{pq}, and have in reality a finite bandwidth. For consistency, such rotating terms should also be ignored elsewhere in Eq. (3.30), and this can be done by making the substitution (3.26), and dropping rotating terms: the result is

$$da = -\frac{i}{\hbar}[a,H_{\rm sys}]dt + \tfrac{1}{2}\gamma \sum_{\substack{l,m\\ l<m}} |c_{lm}|^2\bar{N}(\omega_{ml})(2A_{lm}aA_{ml} - aA_{lm}A_{ml} - A_{lm}A_{ml}a)dt$$
$$+\tfrac{1}{2}\gamma \sum_{\substack{l,m\\ l<m}} |c_{lm}|^2[\bar{N}(\omega_{ml})+1](2A_{ml}aA_{lm} - aA_{ml}A_{lm} - A_{ml}A_{lm}a)dt$$
$$+\sqrt{\gamma}\sum_{\substack{l,m\\ l<m}} \{[a,A_{lm}]c_{lm}dB^\dagger(t,\omega_{ml}) - [a,A_{ml}]c_{lm}^* dB(t,\omega_{ml})\}\ . \tag{3.31}$$

IV. THE MASTER EQUATION

The QSDE'S, is either Ito or Stratonovich form, are often equally intractable for practical consideration. We will now show that the QSDE's are in fact exactly equivalent to an appropriate quantum-mechanical master equation.

A. Description of the density matrix

We consider uncorrelated initial conditions, that is, we assume that the density matrix can be written as a direct product

$$\rho=\rho_s(t_0)\otimes\rho_B(t_0)\ . \tag{4.1}$$

Here, $\rho_s(t_0)$ specifies the initial state of the system variables, and $\rho_B(t_0)$ specifies the initial state of $b_0(\omega)$, and hence of $b_{\rm in}(t)$ for all t in the future of t_0. We assume $\rho_B(t_0)$ is such that for any interval $[t,t']$ in the future of t_0, $B(t,t')$ has the density matrix (3.6).

B. Derivation of the master equation

The mean of any operator $a(t)$ is given by

$$\langle a(t)\rangle = {\rm Tr}_s\,{\rm Tr}_B[a(t)\rho_s(t_0)\otimes\rho_B(t_0)] \tag{4.2}$$
$$={\rm Tr}_s\,{\rm Tr}_B[U(t,t_0)a(t_0)U^\dagger(t,t_0)\rho_s(t_0)\otimes\rho_B(t_0)]\ , \tag{4.3}$$

where $U(t,t_0)$ is the time evolution operator $\exp[-iH(t-t_0)/\hbar]$, with Hamiltonian H as in Eq. (2.1). Using the cyclic property of the trace, we then find

$$\langle a(t)\rangle = {\rm Tr}_s[a(t_0)\hat{\rho}(t)]\ , \tag{4.4}$$

where $\hat{\rho}(t)$ is the time-dependent reduced density matrix, given by

$$\hat{\rho}(t)={\rm Tr}_B[U^\dagger(t,t_0)\rho_s(t_0)\otimes\rho_B(t_0)U(t,t_0)]\ . \tag{4.5}$$

All these results are exact for any system. We now use the QSDE in the Ito form (3.11) [equivalent to the quantum Langevin equation (2.12) provided the input is quantum white noise] to derive the master equation. We note that we can write Eq. (3.11) in the form

$$da(t)=A\{\underline{a}(t)\}dt+G^\dagger\{\underline{a}(t)\}dB(t)+G\{\underline{a}(t)\}dB^\dagger(t) \tag{4.6}$$

[where $\underline{a}(t)$ is the set of all system operators]. Because of the construction of $dB(t),dB^\dagger(t)$ as Ito increments, which means that $dB(t)=B(t+dt)-B(t)$, and the fact that $\underline{a}(t)$ depend only on the past values of $B(t)$, we have

$$\langle da(t)\rangle = \langle A\{\underline{a}(t)\}\rangle dt \tag{4.7}$$

and from (4.4)

$$da(t)={\rm Tr}_s[A\{\underline{a}(t_0)\}\hat{\rho}(t)]dt\ . \tag{4.8}$$

For simplicity and clarity, we now shall use the notation

$$\hat{a}=a(t_0) \tag{4.9}$$

for the operators, which are essentially the Schrödinger-picture operators. Using the form of $A\{\underline{a}(t)\}$ as given in (3.22) and the cyclic property of the trace, we derive

$$\frac{d\langle a(t)\rangle}{dt} = {\rm Tr}_s\left[\hat{a}\left[\frac{i}{\hbar}[\hat{\rho},H_{\rm sys}] + \frac{\gamma}{2}(\bar{N}+1)(2\hat{c}\hat{\rho}\hat{c}^\dagger - \hat{c}^\dagger\hat{c}\hat{\rho} - \hat{\rho}\hat{c}^\dagger\hat{c}) + \frac{\gamma}{2}\bar{N}(2\hat{c}^\dagger\hat{\rho}\hat{c} - \hat{c}\hat{c}^\dagger\hat{\rho} - \hat{\rho}\hat{c}\hat{c}^\dagger)\right]\right]\ . \tag{4.10}$$

However, from (4.4) we have

$$\frac{d\langle a(t)\rangle}{dt} = {\rm Tr}_s\left[a\frac{d\hat{\rho}(t)}{dt}\right]\ . \tag{4.11}$$

Both these equations are valid for any system operator: hence we derive the master equation

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\rho,H_{\rm sys}] + \frac{\gamma}{2}(\bar{N}+1)(2\hat{c}\hat{\rho}\hat{c}^\dagger - \hat{c}^\dagger\hat{c}\hat{\rho} - \hat{\rho}\hat{c}^\dagger\hat{c}) + \frac{\gamma}{2}\bar{N}(2\hat{c}^\dagger\hat{\rho}\hat{c} - \hat{c}\hat{c}^\dagger\hat{\rho} - \hat{\rho}\hat{c}\hat{c}^\dagger) \equiv \hat{L}\hat{\rho}\ . \tag{4.12}$$

(This equation is considered to define $\hat{L}$.) Similarly, the master equation corresponding to the QSDE (3.32) is

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, H_{\text{sys}}] + \frac{\gamma}{2} \sum_{\substack{l,m \\ l<m}} |c_{lm}|^2[\overline{N}(\omega_{ml})+1](2A_{lm}\hat{\rho}A_{ml} - A_{ml}A_{lm}\hat{\rho} - \hat{\rho}A_{ml}A_{lm}) + \frac{\gamma}{2} \sum_{\substack{l,m \\ l<m}} |c_{lm}|^2\overline{N}(\omega_{ml})(2A_{ml}\hat{\rho}A_{lm} - A_{lm}A_{ml}\hat{\rho} - \hat{\rho}A_{lm}A_{ml}) . \tag{4.13}$$

C. Range of validity of master equations

The two kinds of master equation (4.12) and (4.13) obviously have different regions of validity, which depend on the possibility of replacing the term $c(t)b_{\text{in}}^\dagger(t)$ and its Hermitian conjugate by quantum white noise, according to either

$$c(t)b_{\text{in}}^\dagger(t)dt \to c(t)dB^\dagger(t) \tag{4.14}$$

which gives the master equation (4.12) or

$$c(t)b_{\text{in}}^\dagger(t)dt \to \sum_{\substack{l,m \\ l<m}} c_{lm}A_{lm}(t)dB^\dagger(t,\omega_{ml}) \tag{4.15}$$

which gives the master equation (4.13).

The justification of these replacements is a mathematical question; namely, appropriate limits on parameters need to be specified, in which the exact equations corresponding to the original Hamiltonian (2.1) have solutions which approach those of (4.12) or (4.13).

Effectively, the work of Lax[7] has shown in what limits the replacement (4.15) is appropriate, namely, the following.

(i) The energy levels $h\omega_i$ of the system must be nondegenerate, and we must have

$$(\omega_{ml} = \omega_{pq}) \Longrightarrow (m = p \text{ and } l = q) \text{ or } (m = l \text{ and } p = q) . \tag{4.16}$$

(ii) The quantities ω_{ml} must be widely different from each other. Physically, the broadened atomic lines must not overlap, otherwise they will couple to baths which are not entirely independent.

(iii) The coupling of the bath to the system must be weak.

The replacement (4.14) is therefore valid when there is essentially only one frequency in the system. This can arise because of the following.

(i) All levels are equally spaced, and transitions are possibly only from one level to an adjacent level as in the harmonic oscillator, or the two-level atom.

(ii) Because level spacings are equal to each other within the linewidth induced by the damping. For an anharmonic term added to a harmonic oscillator, this means the anharmonicity is less than the damping.

(iii) Although there is a range of frequencies, $\overline{N}(\omega)$ may not change significantly over this range.

D. Time correlation functions

A complete theory must allow us to compute multitime correlation functions directly from the master equation: how to do this is well known, and is usually presented as an approximation to an exact result, and derived by similar approximations to those involved in the master equation. In this section, we will show how the usual formulas are simple to derive directly and without further approximation from the QSDE (3.11).

The only correlation functions which can be computed from the master equation have the structure of a time-ordered product followed by a time-antiordered product, which we shall call multitime-ordered correlation functions, thus

$$\begin{aligned}\langle c_0(s_0)c_1(s_1)\cdots c_m(s_m)a_n(t_n)a_{n-1}(t_{n-1})\cdots a_0(t_0)\rangle &= \mathrm{Tr}_s\,\mathrm{Tr}_B[a_n(t_n)a_{n-1}(t_{n-1})\cdots a_0(t_0)\rho_s \\ &\quad \times\rho_B c_0(s_0)c_1(s_1)\cdots c_m(s_m)] \\ &\equiv G(t_n,t_{n-1},\ldots;s_m,s_{m-1},\ldots s_0) ,\end{aligned} \tag{4.17}$$

where

$$\begin{aligned} &t > t_{n-1} > \cdots > t_0 , \\ &s_m > s_{m-1} > \cdots > s_0 , \end{aligned} \tag{4.18}$$

and ρ_s and ρ_B are evaluated the the earliest of t_0,s_0. Let us assume t_0 is the earliest time and t_n the latest time. Now, in the same way as in Sec. IV B, we can show that

$$\frac{d}{dt_n}G(t_n,t_{n-1},\ldots) = \mathrm{Tr}_s\{a_n(t_0)\hat{L}\,\mathrm{Tr}_B[U^\dagger(t_n,t_0)a_{n-1}(t_{n-1})\cdots\rho_s\otimes\rho_B c_0(s_0)\cdots c_m(s_m)U(t_n,t_0)]\} \tag{4.19}$$

and since this is true for any $a_n(t_0)$, we have shown that

$$\left[\hat{L} - \frac{d}{dt_n}\right]\mathrm{Tr}_B[U^\dagger(t_n,t_0)a_{n-1}(t_{n-1})\cdots\rho_s\otimes\rho_B c_0(s_0)\cdots c_m(s_m)U(t_n,t_0)] = 0 . \tag{4.20}$$

Thus we can now express this quantity in terms of its value when t_n is set equal to the next-latest time in either of the t or s sequences. This must be either t_{n-1} or s_m. If it is s_m, we can write

$$\mathrm{Tr}_B[U^\dagger(t_n,t_0)a_{n-1}(t_{n-1})\cdots\rho_s\otimes\rho_B c_0(s_0)\cdots c_m(s_m)U(t_n,t_0)]$$
$$=\exp[\hat{L}(t_n-s_m)]\{\mathrm{Tr}_B[U^\dagger(s_m,t_0)a_{n-1}(t_{n-1})\cdots\rho_s\otimes\rho_B c_0(s_0)\cdots c_{m-1}(s_{m-1})U(s_m,t_0)]c_m(t_0)\}\ . \quad (4.21)$$

Similarly, if the next time is t_{n-1}, we can then write (4.21) as equal to

$$\exp[\hat{L}(t_n-t_{n-1})]\{a_{n-1}(t_0)\mathrm{Tr}_B[U^\dagger(t_{n-1},t_0)a_{n-2}(t_{n-2})\cdots\rho_s\otimes\rho_B\cdots c_m(s_m)U(t_{n-1},t_0)]\}\ . \quad (4.22)$$

We can similarly show that if the latest time is s_m, then

$$\mathrm{Tr}_B[U^\dagger(s_m,t_0)a_n(t_n)\cdots\rho_s\otimes\rho_B\cdots c_{m-1}(s_{m-1})]$$
$$=\begin{cases}\exp[\hat{L}(s_m-t_n)]\{a_n(t_0)\mathrm{Tr}_B[U^\dagger(t_n,t_0)a_{n-1}(t_{n-1})\cdots\rho_s\otimes\rho_B\cdots c_{m-1}(s_{m-1})U(t_n,t_0)]\} & \text{if } t_n>s_{m-1} \quad (4.23)\\ \exp[\hat{L}(s_m-s_{m-1})]\{\mathrm{Tr}_B[U^\dagger(s_{m-1},t_0)a_n(t_n)\cdots\rho_s\otimes\rho_B\cdots c_{m-2}(s_{m-2})U(s_{m-1},t_0)]c_{m-1}(t_0)\} & \text{if } s_{m-1}>t_n\ . \end{cases}$$
(4.24)

1. Rule for computation of multitime-ordered correlation functions

Proceeding this way, we eventually derive the following rule for the multitime ordered correlation functions.

(1) Order the times in sequence from earliest to latest, and rename them τ_r:

$$\tau_0<\tau_1<\cdots<\tau_{r-1}<\tau_r\ .$$

(2) Define f_r to be the operator corresponding to the time τ_r, but evaluated at time t_0 (i.e., in the Schrödinger picture).

(3) Define a product between any of the f_r and any other expression Y by

$$f_r*Y=f_rY$$

if f_r is one of the a's (i.e., occurs in the time-ordered part),

$$f_r*Y=Yf_r$$

if f_r is one of the c's (i.e., occurs in the time-antiordered part).

(4) Then

$$\langle c_0(s_0)c_1(s_1)\cdots c_m(s_m)a_n(t_n)a_{n-1}(t_{n-1})\cdots a_0(s_0)\rangle$$
$$=\mathrm{Tr}_s[f_r*\exp[\hat{L}(\tau_r-\tau_{r-1})]$$
$$\times(f_{r-1}*\exp[\hat{L}(\tau_{r-1}-\tau_{r-2})]$$
$$\times\{f_{r-2}*\cdots*\exp[\hat{L}(\tau_1-\tau_0)](f_0*\rho_s)\})]\ . \quad (4.25)$$

2. Comments

(1) There are correlation functions which cannot be computed by this formula, namely, those which involve time orderings other than a time-ordered product followed by a time-antiordered product. In practice these do not turn up, for two reasons.

(i) We measure correlation functions of the output, which we will shortly show depend on internal correlation functions of this kind only.

(ii) The quantum theory of measurement seems to generate only this kind of correlation function, i.e., the result of repeated measurements on the same system generates quantities of this kind only.[13,14]

(2) The formula derived here is exactly that derived directly by perturbation methods.[7,14,16]

E. Squeezed white noise as the input signal

An exact theory of quantum stochastic integration can also be developed for the case that the input white noise is *squeezed*, i.e., in which

$$\langle b_{\mathrm{in}}(t)b_{\mathrm{in}}(t')\rangle=M\delta(t-t')\ ,$$
$$\langle b_{\mathrm{in}}^\dagger(t)b_{\mathrm{in}}^\dagger(t')\rangle=M^*\delta(t-t') \quad (4.26)$$

which corresponds to replacing the first of Eq. (3.13) by

$$[dB(t)]^2=M\,dt,\quad [dB^\dagger(t)]^2=M^*dt\ . \quad (4.27)$$

Using these assumptions, we find that the Ito QSDE should be taken in the form

$$da=-\frac{i}{\hbar}[a,H_{\mathrm{sys}}]+\frac{\gamma}{2}(\bar{N}+1)(2c^\dagger ac-ac^\dagger c-c^\dagger ca)dt+\frac{\gamma}{2}\bar{N}(2cac^\dagger-acc^\dagger-cc^\dagger a)dt-\frac{\gamma}{2}M(2c^\dagger ac^\dagger-ac^\dagger c^\dagger-c^\dagger c^\dagger a)dt$$
$$-\frac{\gamma}{2}M^*(2cac-acc-cca)dt-\sqrt{\gamma}[a,c^\dagger]dB(t)+\sqrt{\gamma}[a,c]dB^\dagger(t) \quad (4.28)$$

arising from the relationship between the Ito and Stratonovich integrals now being of the form

$$\mathrm{S}\int_{t_0}^{t} g(t')dB(t') = \mathrm{I}\int_{t_0}^{t} g(t')dB(t') + \tfrac{1}{2}\sqrt{\gamma}\,\overline{N}\int_{t_0}^{t}[g(t'),c(t')]dt' - \tfrac{1}{2}\sqrt{\gamma}M\int_{t_0}^{t}[g(t'),c^{\dagger}(t')]dt' , \tag{4.29a}$$

$$\mathrm{S}\int_{t_0}^{t} dB^{\dagger}(t')g(t') = \mathrm{I}\int_{t_0}^{t} g(t')dB^{\dagger}(t') - \tfrac{1}{2}\sqrt{\gamma}\,\overline{N}\int_{t_0}^{t}[g(t'),c^{\dagger}(t')]dt' + \tfrac{1}{2}\sqrt{\gamma}M^{*}\int_{t_0}^{t}[g(t'),c(t')]dt' . \tag{4.29b}$$

[The Stratonovich integrals with $g(t')$ and increment permuted differ from these by the replacement $\overline{N}\to\overline{N}+1$, corresponding to the commutation relation (2.25) still being true.] The Stratonovich QSDE corresponding to (4.29) is exactly Eq. (3.22) as is expected. The Stratonovich QSDE is independent of the statistics of the incoming field, and in this sense is more general than the Ito QSDE.

The master equation can be derived as in Sec. IV B: it is

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{i}{\hbar}[\rho,H_{\mathrm{sys}}] + \frac{\gamma}{2}(\overline{N}+1)(2c\rho c^{\dagger} - c^{\dagger}c\rho - \rho c^{\dagger}c)\\ &\quad + \frac{\gamma}{2}\overline{N}(2c^{\dagger}\rho c - cc^{\dagger}\rho - \rho cc^{\dagger})\\ &\quad - \frac{\gamma}{2}M(2c^{\dagger}\rho c^{\dagger} - c^{\dagger}c^{\dagger}\rho - \rho c^{\dagger}c^{\dagger})\\ &\quad - \frac{\gamma}{2}M^{*}(2c\rho c - cc\rho - \rho cc) .\end{aligned}$$

This equation can be expected to give the correct description of a system driven by a squeezed noise (which arises from a squeezed vacuum) provided the squeezing is reasonably constant over a bandwidth significantly larger than that expected from the system.

V. RELATIONSHIPS BETWEEN INPUT, OUTPUT, AND INTERNAL CORRELATION FUNCTIONS

We will normally specify the correlation functions of the input, and will wish to compute those of the system and those of the output. From this will arise two principal problems.

First, given the correlation functions of the input, how do we compute those of the system? In certain situations we know the answer. If the QSDE for the system is linear, it can be explicitly solved, and correlations directly calculated. We have already carried out this procedure in a previous paper,[17] and it is also employed by Yurke and Denker[6] in their work on quantum network theory. It does not matter for linear systems what kind of statistics the input has, a direct solution for internal modes and output is possible.

In nonlinear situations we can get a master equation exactly equivalent to the internal QSDE, *provided* the input field consists of ordinary white quantum noise, or squeezed white noise. In practice this is not the case, either because the noise is not white, but has a nonflat spectrum, or because the particular model we have formulated is not exactly valid. However, there *are* many cases where the white-noise approximation is reasonably valid, and the internal modes of these can be treated using the master equation and the numerous techniques available for that. Thus the correlation functions of the internal modes can be calculated in very many cases.

The second consideration is how to compute the correlation functions of the output. We will show that these can be related directly to those of the internal modes, but are not identical in all cases.

A. Output correlation functions

The output correlation functions of most interest are the normally ordered correlation functions of the form

$$\langle b_{\mathrm{out}}^{\dagger}(t_1)b_{\mathrm{out}}^{\dagger}(t_2)\cdots b_{\mathrm{out}}^{\dagger}(t_n)b_{\mathrm{out}}(t_{n+1})\cdots b_{\mathrm{out}}(t_m)\rangle . \tag{5.1}$$

It does not matter what time ordering is chosen in (5.1) since all the $b_{\mathrm{out}}(t_r)$ commute with each other, and all the $b_{\mathrm{out}}^{\dagger}(t_r)$ commute with each other. Non-normally-ordered products can be related to normally ordered products by use of the commutation relations for the output field, which we have shown are the same as those for the input field.

Thus we may consider with complete generality that the correlation function is of the form (5.1) and that the $b_{\mathrm{out}}^{\dagger}(t_r)$ are time *antiordered* (i.e., $t_1<t_2<\cdots<t_n$) and the $b_{\mathrm{out}}(t_r)$ are time ordered (i.e., $t_{n+1}>t_{n+2}>\cdots>t_m$).

We may now substitute for $b_{\mathrm{out}}(t_r)$ by the relation (1.21):

$$b_{\mathrm{out}}(t_r)=\sqrt{\gamma}c(t_r)+b_{\mathrm{in}}(t_r) . \tag{5.2}$$

We note that from (2.26) $c^{\dagger}(t_r)$ will commute with all $b_{\mathrm{in}}^{\dagger}(t_s)$ which occur to the right of $c^{\dagger}(t_r)$, since we have shown $t_r<t_s$. Similarly, $c(t_{r'})$ will commute with all $b_{\mathrm{in}}(t_{s'})$ which occur to the left of $c(t_{r'})$, since we have chosen $t_{r'}>t_{s'}$. Thus we may write the correlation function in a form in which we make the substitution (5.2) in (5.1), and reorder the terms so that

(i) the $c^{\dagger}(t_r)$ are time *antiordered,* and are to the left of all $c(t_r)$;

(ii) the $c(t_{r'})$ are *time ordered;*

(iii) the $b_{\mathrm{in}}^{\dagger}(t_s)$ all stand to the left of all other operators; and

(iv) the $b_{\rm in}(t_{s'})$ all stand to the right of all other operators.

We now consider various cases.

a. Input in the vacuum state. In this case, $b_{\rm in}(t)\rho_{\rm in}=\rho_{\rm in}b^\dagger_{\rm in}(t)=0$, and we derive

$$\langle b^\dagger_{\rm out}(t_1)b^\dagger_{\rm out}(t_2)\cdots b^\dagger_{\rm out}(t_n)b_{\rm out}(t_{n+1})\cdots b_{\rm out}(t_m)\rangle = \langle \tilde T[c^\dagger(t_1)c^\dagger(t_2)\cdots c^\dagger(t_n)]T[c(t_{n+1})\cdots c(t_m)]\rangle \ , \tag{5.3}$$

where $\tilde T$ is the time-antiordered product and T the time-ordered product.

b. Input is in a coherent state. In this case

$$b_{\rm in}(t)\rho_B=\beta(t)\rho_B \ , \quad \rho_B b^\dagger_{\rm in}(t)=\beta^*(t)\rho_B \ , \tag{5.4}$$

where $\beta(t)$ is the coherent field amplitude; a c-number. In this case we may replace $b,b^\dagger$ by β,β^*, respectively, and since β and β^* are mere numbers, reorder them so that

$$\langle b^\dagger_{\rm out}(t_1)b^\dagger_{\rm out}(t_2)\cdots b^\dagger_{\rm out}(t_n)b_{\rm out}(t_{n+1})\cdots b_{\rm out}(t_m)\rangle = \langle \tilde T\{[\sqrt{\gamma}c^\dagger(t_1)+\beta^*(t_1)]\cdots\}T\{[\sqrt{\gamma}c(t_{n+1})+\beta(t_{n+1})]\cdots[\sqrt{\gamma}c(t_m)+\beta(t_m)]\}\rangle \ . \tag{5.5}$$

c. The input is quantum white noise. By carrying out the procedure in Sec. V A we are left with the problem of evaluating times like $\langle c^\dagger(t_1)c^\dagger(t_2)b_{\rm in}(s_1)b_{\rm in}(s_2)\rangle$ when the density matrix is not coherent or vacuum, but represents a more general state. The solution of this problem for arbitrary statistics of the input is rather difficult, but we can give a way of doing this calculation in the case that the input is quantum white noise (clearly, this will also allow us to carry out the computation when the input also has an additional coherent part). This requires us, more generally, to calculate terms like $\langle a_1(t)a_2(t')dB(s)dB(s')\rangle$ where $a_1(t)$ and $a_2(t)$ obey a single QSDE of the form (Ito) (3.11) which we can write in an abbreviated form

$$da = A(a(t))dt + G^\dagger(\underline{a}(t))dB(t)+G(a(t))dB^\dagger(t) \ . \tag{5.6}$$

[We shall only treat the case where the simple master equation (4.12) is relevant. The more general situation of Lax's master equation (4.13) is essentially a many-input version of the same procedure; one input and one output being available for each frequency band around the ω_{ml}.] For simplicity, let us consider first the evaluation of $\langle a(t')dB(t)\rangle$. If $t>t'$, then $dB(t)$ is independent of $a(t')$, and so we deduce

$$\langle a(t')dB(t)\rangle = 0, \quad t>t' \ . \tag{5.7}$$

For $t<t'$, we argue as follows. We discretize time and define

$$a_0=a(t), \quad a_n=a(t_n) \ , \quad A_n=A(\underline{a}_n), \quad G_n=G(\underline{a}_n) \ , \tag{5.8}$$

so that (because we use the Ito form of the QSDE)

$$a_n=a_{n-1}+a_{n-1}\Delta t_n+G^\dagger_{n-1}\Delta B_n+G_{n-1}\Delta B^\dagger_n \ . \tag{5.9}$$

To solve the QSDE, we repeatedly use (5.9) starting with the initial condition a_0, and we can eventually write

$$a_n=\sum_{r,s} a_n^{r,s}\{\Delta B_1^\dagger\}^r\{\Delta B_1\}^s \ , \tag{5.10}$$

where the coefficients depend on a_0 and $\Delta B^\dagger_m,\Delta B_{m'}$, for $1\le m$, $m'\le n$. The discretized form of $\langle a(t')dB(t)\rangle$ is (to lowest order in Δt)

$$\langle a_n\,\Delta B_1\rangle = \langle a_n^{1,0}\rangle\langle \Delta B_1^\dagger\Delta B_1\rangle = \bar N\,\Delta t_1\,\langle a_n^{1,0}\rangle \ . \tag{5.11}$$

We now need an expression for $\langle a_n^{1,0}\rangle$. Suppose we modify the QSDE (5.6) by the substitution

$$dB(t)\to dB(t)+\epsilon(t)dt \ , \quad dB^\dagger(t)\to dB^\dagger(t)+\epsilon^*(t)dt \ , \tag{5.12}$$

where $\epsilon(t)$ is a classical c-number driving field. Then discretizing again, it is clear that

$$\langle a_n^{1,0}\rangle = \frac{1}{\Delta t_1}\left\langle \frac{\partial}{\partial\epsilon_1^*}a_n^\epsilon\right\rangle_{\epsilon=0} \ , \tag{5.13}$$

and in a continuum notation this becomes

$$\langle a(t')dB(t)\rangle = \left[\bar N\frac{\delta}{\delta\epsilon^*(t)}\langle a^\epsilon(t')\rangle\right]_{\epsilon=0} \ . \tag{5.14}$$

We now evaluate this expression. We note that from the master equation (4.12)

$$\langle a^\epsilon(t')\rangle = \langle a\rho^\epsilon(t')\rangle = \left\langle aT\left[\exp\left[\int_t^{t'}\hat L^\epsilon(s)ds\right]\right]\rho(t)\right\rangle \tag{5.15}$$

and, discretizing, this can be written

$$\langle a_n^\epsilon\rangle = \langle a(1+\hat L_n^\epsilon\Delta t_n)\times(1+\hat L^\epsilon_{n-1}\Delta t_{n-1})\cdots(1+\hat L^\epsilon_1\Delta t_1)\rho(t)\rangle \ . \tag{5.16}$$

But only the term involving L_1^ϵ depends on ϵ_1^*, and, in fact, the coefficient of ϵ_1^* in L_1^ϵ is $\sqrt{\gamma}[c,\rho]$, so that

$$\frac{\partial}{\partial\epsilon_1^*}\langle a_n^\epsilon\rangle = \langle a(1+L_n^\epsilon\Delta t_n)\cdots(1+L_2^\epsilon\Delta t_2)\times\{-\sqrt{\gamma}[c,\rho(t)]\}\rangle \tag{5.17}$$

and we then find

$$\left[\frac{\delta}{\delta\epsilon^*(t)}\langle a^{\epsilon}(t')\rangle\right]_{\epsilon=0}=\sqrt{\gamma}\langle a\exp[\hat{L}(t'-t)][c,\rho(t)]\rangle$$
$$=\sqrt{\gamma}\langle[a(t'),c(t)]\rangle\ . \quad (5.18)$$

Thus, we find the general expression

$$\langle a(t')dB(t)\rangle=\sqrt{\gamma}\bar{N}\,dt\,u(t'-t)\langle[a(t')c,(t)]\rangle\ . \quad (5.19)$$

We therefore derive that

$$\langle a(t')b_{\rm in}(t)\rangle=\sqrt{\gamma}\,\bar{N}u(t'-t)\langle[a(t'),c(t)]\rangle$$

and similarly

$$\langle b_{\rm in}^{\dagger}(t)a(t')\rangle=\sqrt{\gamma}\,\bar{N}u(t'-t)\langle[c^{\dagger}(t),a(t')]\rangle\ ,$$
$$\langle a(t')b_{\rm in}^{\dagger}(t)\rangle=\sqrt{\gamma}(\bar{N}+1)u(t'-t)\langle[c^{\dagger}(t),a(t')]\rangle\ ,$$
$$\langle b_{\rm in}(t)a(t')\rangle=\sqrt{\gamma}(\bar{N}+1)u(t'-t)\langle[a(t'),c(t)]\rangle \quad (5.20)$$

and we note that these are consistent with the commutation relations (2.26). We can now use the results of Sec. II C to derive

$$\langle b_{\rm out}(t)b_{\rm out}(t')\rangle=\gamma(\bar{N}+1)\langle T[c(t)c(t')]\rangle-\gamma\bar{N}\langle\tilde{T}[c(t)c(t')]\rangle\ ,$$
$$\langle b_{\rm out}^{\dagger}(t)b_{\rm out}(t')\rangle=\gamma(\bar{N}+1)\langle c^{\dagger}(t)c(t')\rangle-\gamma\bar{N}\langle c(t')c^{\dagger}(t)\rangle+\gamma\bar{N}\delta(t-t')\ , \quad (5.21)$$
$$\langle b_{\rm out}^{\dagger}(t)b_{\rm out}^{\dagger}(t')\rangle=\gamma(\bar{N}+1)\langle\tilde{T}[c^{\dagger}(t)c^{\dagger}(t')]\rangle-\gamma\bar{N}\langle T[c^{\dagger}(t)c^{\dagger}(t')]\rangle\ .$$

The procedure can also be applied to the situation of squeezed white noise for the input.

B. High-order correlation functions

We consider a term of the form

$$\langle a(t)dB(s)dB(s')\rangle\ . \quad (5.22)$$

If either s or s' is greater then t, then this is clearly zero. If s and s' are different, it is not difficult to retrace the argument above to show that

$$\langle a(t)dB(s)dB(s')\rangle=\left[\frac{\delta^2}{\delta\epsilon^*(s)\delta\epsilon^*(s')}\langle a^{\epsilon}(t)\rangle\right]_{\epsilon=0}\bar{N}^2\,ds\,ds' \quad (5.23)$$

so that

$$\langle a(t)b_{\rm in}(s)b_{\rm in}(s')\rangle=\bar{N}^2\left[\frac{\delta^2}{\delta\epsilon^*(s)\delta\epsilon^*(s')}\langle a^{\epsilon}(t)\rangle\right]_{\epsilon=0} \quad (5.24)$$

and carrying out the same process as was used previously to evaluate the first-order functional derivative, we find

$$\langle a(t)b_{\rm in}(s)b_{\rm in}(s')\rangle=\begin{cases}\bar{N}^2\langle ae^{\hat{L}(t-s)}[\sqrt{\gamma}c,e^{\hat{L}(s-s')}[\sqrt{\gamma}c,\rho(s')]]\rangle & \text{if } s>s' \text{ and } t>s\\ 0 \ \text{ if } s \text{ or } s'>t\ .\end{cases} \quad (5.25)$$

Hence

$$\langle a(t)b_{\rm in}(s)b_{\rm in}(s')\rangle=\begin{cases}\gamma\bar{N}^2\langle[[a(t),c(s)],c(s')]\rangle, & t>s>s'\\ \gamma\bar{N}^2\langle[[a(t),c(s')],c(s)]\rangle, & t>s'>s\\ 0, & s>t \text{ or } s'>t\ .\end{cases} \quad (5.26)$$

C. Even-higher-order correlation functions

We now consider a term of the form

$$\langle a_1(t)a_2(t')b_{\rm in}(s)b_{\rm in}(s')\rangle \quad (5.27)$$

and in this case a similar procedure gives a variety of answers depending on the time ordering:

$$\langle a_1(t)a_2(t')b_{\rm in}(s)b_{\rm in}(s')\rangle=\begin{cases}\gamma\bar{N}^2\langle[[a_1(t)a_2(t'),c(s)]c(s')]\rangle, & t>t'>s>s' & (5.28)\\ \gamma\bar{N}^2\langle[[a_1(t)a_2(t'),c(s')],c(s)]\rangle, & t>t'>s'>s & (5.29)\\ \gamma\bar{N}^2\langle[[a_1(t),c(s)]a_2(t),c(s')]\rangle, & t>s>t'>s' & (5.30)\\ \gamma\bar{N}^2\langle[[a_1(t),c(s')]a_2(t),c(s)]\rangle, & t>s'>t'>s & (5.31)\\ \gamma\bar{N}^2\langle[[a_1(t),c(s)],c(s')]a_2(t')\rangle, & t>s>s'>t' & (5.32)\\ \gamma\bar{N}^2\langle[[a_1(t),c(s)],c(s')]a_2(t')\rangle, & t>s'>s>t' & (5.33)\\ 0, & s>t \text{ or } s'>t\ . & (5.34)\end{cases}$$

VI. SUMMARY AND CONCLUSIONS

We have drawn together in this paper a number of points of view which have until now been somewhat detached from each other. The literature on quantum Langevin equations has until now been rather sparse and unsystematic, and the master equation, as an alternative treatment of quantum stochastic processes, has not to our knowledge been brought into a direct equivalence with the Langevin approach. The work on quantum stochastic processes, most notably presented in the both by Davies[13] and summarized in Spohn's review,[14] has proceeded essentially from the master-equation point of view, as has the theory of measurement developed with it. These theories are rigorously formulated, though of course are nevertheless no more valid than the hypotheses assumed as far as physics is concerned. We hope that the work in this paper does not give a false sense of absolute correctness. For the systems chosen, and if the input state is quantum white noise, our results are exact, and presumably can be rigorously proved if necessary. But the model Hamiltonians are themselves only approximations to the real world, and depend largely on the rotating-wave approximation; and quantum white noise itself can only be an approximation to the real state of some optical input. As is well known, this will also depend on the system being of narrow bandwidth and operating at a high frequency, so that the variation of the quantum-noise spectral density over the region of interest is small.

However, it was not our aim in this paper to investigate the relationships between real systems and our approximate systems. Rather, the principal aims have been the following.

(i) To codify clearly the relation between input, internal motion, and output in a class of well-defined systems.

(ii) To codify precisely the relation between QSDE, master-equation, and multitime-ordered correlation functions, and present these as a complete mathematical entity.

(iii) To show how to deal with at least some kinds of nonwhite and noncoherent inputs, i.e., squeezed white noise.

This is the first in a planned series of papers. In fact the problem of squeezing in the output of a parametric amplifier system has already been treated by this formalism[17] but the derivations given there are heuristic only, so that this paper provides a solid foundation for those calculations. The results of Sec. V, in particular, have also been applied to the calculation of output squeezing for other nonlinear optical systems,[18] including dispersive optical bistability and second-harmonic generation.

The next paper in the series will deal with a field theory of input and output, taking realistic Hamiltonians, and explicitly identifying the in and out operators with specific quantum fields. That paper will also deal more precisely with the approximations necessary to obtain the kind of theory presented here from a real, and hence more intricate system.

---

[1]See, for example, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Evanston, 1961).

[2]C. M. Caves, Phys. Rev. D **26**, 1817 (1982).

[3]H. J. Kimble, M. Dagenais, and L. Mandel, Phys. Rev. A **18**, 201 (1978).

[4]H. J. Carmichael and D. F. Walls, J. Phys. B **9**, 143 (1976).

[5]C. M. Gardiner and C. M. Savages, Opt. Commun. **50**, 173 (1984).

[6]B. Yurke and J. S. Denker, Phys. Rev. A **29**, 1419 (1984).

[7]M. Lax, Phys. Rev. **145**, 110 (1966).

[8]W.H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

[9]W. Weidlich, H. Risken, and H. Haken, Z. Phys. **201**, 369 (1967); Z. Phys. **204**, 223 (1967).

[10]F. Haake, in *Quantum Statistics in Optics and Solid-State Physics,* Vol. 66 of *Springer Tracts in Modern Physics* (Springer, Berlin, 1973), p. 98.

[11]G. S. Agarwal, in *Quantum Optics,* Vol. 70 of *Springer Tracts in Modern Physics* (Springer, Berlin, 1974).

[12]M. Sargent, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addison-Wesley, Reading, Mass., 1974).

[13]E. B. Davies, *Quantum Theory of Open Systems* (Academic, London, 1976).

[14]H. Spohn, Rev. Mod. Phys. **52**, 569 (1980).

[15]H. Haken, *Light and Matter Ic,* Band 25, Teil 2 of *Handbuch der Physik* (Springer, Berlin, 1970).

[16]C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[17]M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).

[18]M. J. Collett and D. F. Walls, Phys. Rev. A (to be published).