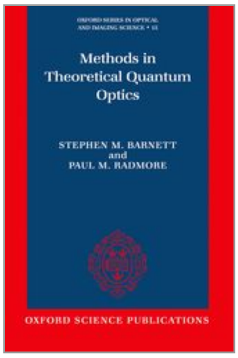

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(p.225) Appendix 2 THE DIRAC DELTA FUNCTION

The Dirac delta function is not a function in the usual sense of being defined for values of its argument, but is a generalized function in that it is defined by a limiting procedure and only has meaning within an integral. It is often written and manipulated without reference to an integral but it should be remembered that results so obtained are strictly prescriptions for manipulating integrals containing delta functions.

The delta function $\delta(x)$ may be defined as the limit as $\epsilon \to 0$ of a normalized function $\phi(x, \epsilon)$ of x and ϵ which is symmetric about its main or only peak at $x = 0$. Such functions tend to zero in the limit $\epsilon \to 0$ for all *non-zero* values of x, and hence, since the function is normalized to unit area, the value at $x = 0$ must tend to infinity. Examples of suitable functions are $\epsilon/[\pi(x^2 + \epsilon^2)]$, $(\pi\epsilon)^{-1/2} \exp(-x^2/\epsilon)$, and $(\pi x)^{-1} \sin(x/\epsilon)$. A delta function $\delta(x - a)$ peaked at $x = a$ is simply obtained by replacing x by $x - a$ in the function and then taking the limit $\epsilon \to 0$. The integral of the delta function is, for $p < q$,

(A2.1)
$$\int_p^q \delta(x-a)\,\mathrm{d}x = \begin{cases} 1 & \text{for } p < a < q, \\ \frac{1}{2} & \text{for } p = a \text{ or } q = a, \\ 0 & \text{otherwise,} \end{cases}$$

where the value unity arises from the normalization and the value $\frac{1}{2}$ is obtained because only one-half of the symmetric function lies within the range of integration. The Heaviside unit step function $H(x - a)$ can then be defined using (A2.1) as

(A2.2)
$$H(x-a) = \int_{-\infty}^{x} \delta(x'-a)\,\mathrm{d}x' = \begin{cases} 0 & \text{for } x < a, \\ \frac{1}{2} & \text{for } x = a, \\ 1 & \text{for } x > a. \end{cases}$$

The important property of $\delta(x-a)$ is the so-called sifting property, which states that if $f(x)$ is a function for which $f(a)$ is defined then

(A2.3)
$$\int_p^q \delta(x-a) f(x)\,\mathrm{d}x = \begin{cases} f(a) & \text{for } p < a < q, \\ \frac{1}{2} f(a) & \text{for } p = a \text{ or } q = a, \\ 0 & \text{otherwise.} \end{cases}$$

This result may be used more generally: if $f(x)$ is itself a generalized function then (A2.3) is meaningful if $f(a)$ is meaningful. For example,

(A2.4)
$$\int_{-\infty}^{\infty} \delta(x-a)\delta(x-b)\,\mathrm{d}x = \delta(a-b)$$

(p.226) within an integral over either a or b. Comparing (A2.1) with (A2.3), we see that

(A2.5)
$$\int_p^q \delta(x-a) f(x)\,\mathrm{d}x = \int_p^q \delta(x-a) f(a)\,\mathrm{d}x$$

and hence we can write

(A2.6)
$$\delta(x-a) f(x) = \delta(x-a) f(a)$$

in the sense that, as before, equality holds when either side of (A2.6) lies within an integral.

The Fourier transform of $\delta(x)$ can be calculated using the sifting property (A2.3) as

(A2.7)
$$\mathscr{F}\{\delta(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x) \exp(\pm \mathrm{i}kx)\,\mathrm{d}x = \frac{1}{2\pi}.$$

Hence a representation of $\delta(x)$ obtained from the inverse transform of $(2\pi)^{-1}$ is

(A2.8)
$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(\mp \mathrm{i}kx)\,\mathrm{d}k.$$

A simple change of variable then shows that

(A2.9)
$$\delta(bx) = \frac{1}{|b|}\,\delta(x).$$

The extension of the definition of the delta function into two or more dimensions is achieved by considering the product of individual delta functions, one for each dimension. For example, if $\mathbf{k} = (k_x, k_y, k_z)$ and $\mathbf{k}' = (k'_x, k'_y, k'_z)$ are two vectors then we define the three-dimensional delta function $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ as

(A2.10)
$$\delta^{(3)}(\mathbf{k} - \mathbf{k}') = \delta(k_x - k'_x)\,\delta(k_y - k'_y)\,\delta(k_z - k'_z).$$

Two-dimensional delta functions are useful in problems involving functions of complex variables, so that for a complex variable $\xi = \xi_\mathrm{r} + \mathrm{i}\,\xi_\mathrm{i}$, we define

(A2.11)
$$\delta^{(2)}(\xi) = \delta(\xi_\mathrm{r})\,\delta(\xi_\mathrm{i}) = \delta^{(2)}(\xi^*).$$

Considering the one-dimensional delta functions in (A2.11) to be the limit of Gaussians leads to the identification

(A2.12)
$$\begin{aligned}\delta(\xi_\mathrm{r})\,\delta(\xi_\mathrm{i}) &= \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon}\exp\left[-(\xi_\mathrm{r}^{\,2} + \xi_\mathrm{i}^{\,2})/\epsilon\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon}\exp\left(-|\xi|^2/\epsilon\right),\end{aligned}$$

with $\delta^{(2)}(\xi)$ (see (4.4.18)). It follows from (A2.8) and (A2.11) that a representation of $\delta^{(2)}(\xi)$ is

(A2.13)
$$\begin{aligned}\delta^{(2)}(\xi) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}(2\alpha_\mathrm{r})\,\mathrm{d}(2\alpha_\mathrm{i})\exp(-2\mathrm{i}\,\alpha_\mathrm{r}\,\xi_\mathrm{i} + 2\mathrm{i}\,\alpha_\mathrm{i}\,\xi_\mathrm{r}) \\ &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\mathrm{d}^2\alpha\,\exp(\alpha\xi^* - \alpha^*\xi),\end{aligned}$$

 where $\mathrm{d}^2\alpha = \mathrm{d}\,\alpha_\mathrm{r}\,\mathrm{d}\,\alpha_\mathrm{i}$ and integration is implied over the whole of the complex α-plane.

Consider the integral

(A2.14)
$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi(x,\epsilon)}{\mathrm{d}x}f(x)\,\mathrm{d}x = [\phi(x,\epsilon)f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\phi(x,\epsilon)f'(x)\,\mathrm{d}x, \quad \text{(A2.}$$

where the limit as $\epsilon \to 0$ of $\phi(x, \epsilon)$ is $\delta(x)$. Assuming the integrated term in (A2.14) is zero, as it will be for all functions $f(x)$ for which the first integral exists, then in the limit $\epsilon \to 0$ the right-hand side of (A2.14) becomes $-f'(0)$, using (A2.3). This gives rise to the symbol $\delta'(x)$ which we call the first derivative of the delta function, given by the limit $\delta'(x) = \lim_{\epsilon \to 0} \phi'(x, \epsilon)$ with the property

(A2.15) $$\int_{-\infty}^{\infty} \delta'(x) f(x)\, \mathrm{d}x = -f'(0),$$

provided $f'(0)$ exists. We can extend this to higher-order derivatives of $\delta(x)$, written $\mathrm{d}^m\delta(x)/\mathrm{d}x^m$, with the properties

(A2.16) $$\int_{-\infty}^{\infty} \frac{\mathrm{d}^m\delta(x)}{\mathrm{d}x^m} f(x)\, \mathrm{d}x = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^m f(x)\bigg|_{x=0}.$$

An alternative representation of $\delta'(x)$ can be found by considering

(A2.17)
$$\int_{-\infty}^{\infty} \delta'(x) x g(x)\, \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}x}[x g(x)]\bigg|_{x=0} = -g(0),$$

using (A2.15), provided $g(0)$ exists. Further, using the sifting property (A2.3) with $f(x) = -g(x)$, we have

(A2.18)
$$-g(0) = \int_{-\infty}^{\infty} \delta(x)[-g(x)]\, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{-\delta(x)}{x} x g(x)\, \mathrm{d}x.$$

Comparing (A2.17) with (A2.18) leads to the identification

(A2.19) $$\delta'(x) = -\frac{\delta(x)}{x}$$

with the usual understanding that this should be interpreted as a rule for manipulating integrals containing delta functions. The two-dimensional extension of (A2.16) for $\delta^{(2)}(\xi)$ and $f(\xi)$, where ξ is a complex variable, is

(A2.20)
$$\int_{-\infty}^{\infty} \mathrm{d}^2\xi\, f(\xi) \left(\frac{\partial}{\partial \xi}\right)^m \left(\frac{\partial}{\partial \xi^*}\right)^n \delta^{(2)}(\xi) = \left(-\frac{\partial}{\partial \xi}\right)^m \left(-\frac{\partial}{\partial \xi^*}\right)^n f(\xi)\bigg|_{\xi=0}.$$