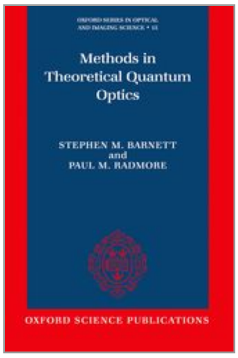

Methods in Theoretical Quantum Optics
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Print publication date: 2002
Print ISBN-13: 9780198563617
Published to Oxford Scholarship Online: January 2010
DOI: 10.1093/acprof:oso/9780198563617.001.0001

(p.236) Appendix 5 OPERATOR ORDERING THEOREMS

In Chapter 3, we derived a number of theorems relating different orderings of the exponential function of operators. Here we summarize these theorems and give some alternative forms and more general expressions.

For two operators $\hat{A}$ and $\hat{B}$ which commute with their commutator $[\hat{A}, \hat{B}]$ we have

(A5.1)
$$\exp[\theta(\hat{A}+\hat{B})] = \exp(\theta\hat{A})\exp(\theta\hat{B})\exp\left(-\tfrac{1}{2}\theta^2[\hat{A},\hat{B}]\right).$$

Important examples of this are

(A5.2)
$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})\exp\left(-|\alpha|^2/2\right)$$
$$= \exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^\dagger)\exp\left(|\alpha|^2/2\right),$$

(A5.3) $$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp[\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)],$$

and the continuum operator analogue of these given by

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(A5.4)

$$\hat{D}[\alpha(\omega)] = \exp\left(\int d\omega[\alpha(\omega)\hat{b}^\dagger(\omega) - \alpha^*(\omega)\hat{b}(\omega)]\right)$$

$$= \exp\left(\int d\omega\, \alpha(\omega)\hat{b}^\dagger(\omega)\right) \exp\left(-\int d\omega\, \alpha^*(\omega)\hat{b}(\omega)\right)$$

$$\times \exp\left(-\tfrac{1}{2}\int d\omega |\alpha(\omega)|^2\right)$$

$$= \exp\left(-\int d\omega\, \alpha^*(\omega)\hat{b}(\omega)\right) \exp\left(\int d\omega\, \alpha(\omega)\hat{b}^\dagger(\omega)\right)$$

$$\times \exp\left(\tfrac{1}{2}\int d\omega |\alpha(\omega)|^2\right)$$

and

(A5.5)

$$\hat{D}[\alpha(\omega)]\hat{D}[\beta(\omega)] = \hat{D}[\alpha(\omega) + \beta(\omega)]$$

$$\times \exp\left(\tfrac{1}{2}\int d\omega[\alpha(\omega)\beta^*(\omega) - \alpha^*(\omega)\beta(\omega)]\right).$$

The exponential function of the number operator is related to its normal ordered form by

(A5.6) $$\exp(\theta \hat{a}^\dagger \hat{a}) = :\exp\{[\exp(\theta) - 1]\hat{a}^\dagger \hat{a}\}:$$

(p.237) with the continuum operator generalization

(A5.7)

$$\exp\left(\int d\omega\, \theta(\omega)\hat{b}^\dagger(\omega)\hat{b}(\omega)\right) = :\exp\left(\int d\omega \{\exp[\theta(\omega)] - 1\}\hat{b}^\dagger(\omega)\hat{b}(\omega)\right):$$

There is an antinormal ordered analogue of (A5.6) given by

(A5.8) $$\exp(\theta \hat{a}\hat{a}^\dagger) = \vdots \exp\{[1 - \exp(-\theta)]\hat{a}^\dagger \hat{a}\} \vdots$$

but there is no antinormal ordered continuum generalization.

For two operators $\hat{A}$ and $\hat{B}$ with commutator $[\hat{A}, \hat{B}] = -\hat{A}$, we have

(A5.9)

$$\exp[\theta(\hat{A} + \hat{B})] = \exp(\theta\hat{B}) \exp\{[1 - \exp(-\theta)]\hat{A}\}$$

$$= \exp\{[\exp(\theta) - 1]\hat{A}\} \exp(\theta\hat{B}).$$

This result may also be applied to superoperators as in Section 5.6.

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The angular momentum operators $\hat{J}_3, \hat{J}_+$, and $\hat{J}_-$ satisfy the commutation relations $[\hat{J}_3, \hat{J}_\pm] = \pm\hat{J}_\pm$ and $[\hat{J}_+, \hat{J}_-] = 2\hat{J}_3$. We have

(A5.10)

$$\exp\left[\mathrm{i}\theta\left(\hat{J}_+ + \hat{J}_-\right)\right] = \exp\left[\mathrm{i}(\tan\theta)\hat{J}_+\right]\exp\left[-\ln(\cos^2\theta)\hat{J}_3\right]\exp\left[(\mathrm{i}\tan\theta)\hat{J}_-\right.$$

which may be generalized to

(A5.11)

$$\exp\left(\lambda_+\hat{J}_+ + \lambda_-\hat{J}_- + \lambda_3\hat{J}_3\right) = \exp\left(\Lambda_+\hat{J}_+\right)\exp\left[(\ln\Lambda_3)\hat{J}_3\right]\exp\left(\Lambda_-\hat{J}_-\right)$$
$$= \exp\left(\Lambda_-\hat{J}_-\right)\exp\left[-(\ln\Lambda_3)\hat{J}_3\right]\exp\left(\Lambda_+\hat{J}_+\right),$$

where

(A5.12) $$\Lambda_3 = \left(\cosh\alpha - \frac{\lambda_3}{2\alpha}\sinh\alpha\right)^{-2},$$

(A5.13) $$\Lambda_\pm = \frac{2\lambda_\pm\sinh\alpha}{2\alpha\cosh\alpha - \lambda_3\sinh\alpha},$$

and

(A5.14) $$\alpha^2 = \tfrac{1}{4}\lambda_3^2 + \lambda_+\lambda_-.$$

These theorems can be applied to a pair of field modes with annihilation operators $\hat{a}$ and $\hat{b}$ by making the identifications

(A5.15) $$\hat{J}_3 = \tfrac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}),$$

(A5.16) $$\hat{J}_+ = \hat{a}^\dagger\hat{b} = \hat{J}_-^\dagger.$$

(p.238) The operators $\hat{K}_3, \hat{K}_+$, and $\hat{K}_-$ satisfy the commutation relations $[\hat{K}_3, \hat{K}_\pm] = \pm\hat{K}_\pm$ and $[\hat{K}_+, \hat{K}_-] = -2\hat{K}_3$. We have

(A5.17)

$$\exp\left[\mathrm{i}\theta\left(\hat{K}_+ + \hat{K}_-\right)\right] = \exp\left[\mathrm{i}(\tanh\theta)\hat{K}_+\right]\exp\left[-\ln(\cosh^2\theta)\hat{K}_3\right]\exp\left[\mathrm{i}(\tanh\right.$$

which may be generalized to

(A5.18)

$$\exp\left(\gamma_+\hat{K}_+ + \gamma_-\hat{K}_- + \gamma_3\hat{K}_3\right) = \exp\left(\Gamma_+\hat{K}_+\right)\exp\left[(\ln\Gamma_3)\hat{K}_3\right]\exp\left(\Gamma_-\hat{K}_-\right)$$
$$= \exp\left(\Gamma_-\hat{K}_-\right)\exp\left[-(\ln\Gamma_3)\hat{K}_3\right]\exp\left(\Gamma_+\hat{K}_+\right)$$

where

(A5.19) $$\Gamma_3 = \left(\cosh\beta - \frac{\gamma_3}{2\beta}\sinh\beta\right)^{-2},$$

(A5.20) $$\Gamma_\pm = \frac{2\gamma_\pm \sinh\beta}{2\beta\cosh\beta - \gamma_3\sinh\beta},$$

and

(A5.21) $$\beta^2 = \tfrac{1}{4}\gamma_3^2 - \gamma_+\gamma_-.$$

These theorems can be applied either to a single field mode by making the identifications

(A5.22) $$\hat{K}_3 = \tfrac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger),$$

(A5.23) $$\hat{K}_+ = \tfrac{1}{2}\hat{a}^{\dagger 2} = \hat{K}_-^\dagger,$$

or to a pair of modes by making the identifications

(A5.24) $$\hat{K}_3 = \tfrac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger),$$

(A5.25) $$\hat{K}_+ = \hat{a}^\dagger\hat{b}^\dagger = \hat{K}_-^\dagger.$$

Two operators are equivalent if their matrix elements between any two basis states are equal, for all possible pairs of basis states. This can be used to derive ordering theorems. We illustrate this by obtaining (A5.6) using the number state and coherent state bases. As in Chapter 3, let

(A5.26) $$\exp(\theta\hat{a}^\dagger\hat{a}) = :\exp\left[p(\theta)\hat{a}^\dagger\hat{a}\right]:.$$

(p.239) The number state matrix elements of each side of (A5.26) are

(A5.27) $$\langle n|\exp(\theta\hat{a}^\dagger\hat{a})|n'\rangle = \exp(\theta n)\delta_{nn'}$$

and

(A5.28) $$\langle n|:\exp\left[p(\theta)\hat{a}^\dagger\hat{a}\right]:|n'\rangle = \sum_{l=0}^{n} C_l^n[p(\theta)]^l\,\delta_{nn'} = [1+p(\theta)]^n\,\delta_{nn'},$$

using (3.4.1). It follows immediately that $p(\theta) = \exp(\theta) - 1$. Alternatively we can use the coherent state basis to establish this result. It is sufficient to equate only the diagonal matrix elements in this basis since, apart from a normalization factor $\exp(-|\alpha|^2/2)$, $|\alpha\rangle$ depends only on α while $\langle\alpha|$ depends only on α^*, and these are treated as two independent variables. The diagonal coherent state matrix elements of each side of (A5.26) are

(A5.29)

$$\langle \alpha | \exp(\theta \hat{a}^{\dagger}\hat{a}) | \alpha \rangle = \langle \alpha | \alpha \exp(\theta) \rangle \exp\left\{\tfrac{1}{2}|\alpha|^2[\exp(2\theta) - 1]\right\}$$
$$= \exp\left\{-|\alpha|^2[1 - \exp(\theta)]\right\},$$

using (3.6.32) and (3.6.24), and

(A5.30) $$\langle \alpha | :\exp\left[p(\theta)\hat{a}^{\dagger}\hat{a}\right]: | \alpha \rangle = \exp\left[|\alpha|^2 p(\theta)\right],$$

using $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. Equating (A5.29) and (A5.30) again gives $p(\theta) = \exp(\theta) - 1$.