Applied Physics 190c: Homework #3

(Dated: April 27, 2017)

Due: Friday, May 5th.

1. **Reading**

   Chapter 4 in "Methods in Theoretical Quantum Optics" [1], Chapter 3 and 4 in "Statistical Methods of Quantum Optics" [2].

2. **(60 points)** ***P*, *W*, and *Q* phase-space representations**

   One can define the different phase-space representations using the p-ordered characteristic function,

$$\chi(\xi, p) = \mathrm{Tr}\left[\hat{\rho}\hat{D}(\xi)\right]\exp(p|\xi|^2/2), \tag{1}$$

   where $\hat{D}(\xi) = \exp(\xi\hat{a}^\dagger - \xi^*\hat{a})$ is the displacement operator with complex displacement amplitude ξ.

   (a) The values $p = 1$, 0, and -1 correspond to a characteristic function of the normal, symmetric, and antinormal ordered version of the displacement operator. Show this.

   (b) Show that the expectation value of the p-ordered product of $\left(\hat{a}^\dagger\right)^m$ and $\hat{a}^n$ is given by,

$$\left\langle \left(\hat{a}^\dagger\right)^m a^n \right\rangle_p = (\partial/\partial\xi)^m(-\partial/\partial\xi^*)^n\chi(\xi, p)|_{\xi=0}. \tag{2}$$

   Note that ξ and ξ^* are to be treated as independent variables with respect to differentiation. This is possible due to the fact that $(\partial/\partial\xi)(\xi^*) \equiv (1/2)(\partial/\partial\xi_r - i\partial/\partial\xi_i)(\xi_r - i\xi_i) = 0$, where ξ_r and ξ_i are the real and imaginary components of ξ, respectively.

   (c) Show that one can rewrite the characteristic function in terms of quadrature operators $\hat{x}_\lambda$ and $\hat{x}_{\lambda+\pi/2}$,

$$\chi(\xi, p) = \mathrm{Tr}\left\{\hat{\rho}\exp\left[-i(\xi_\lambda\hat{x}_\lambda + \xi_{\lambda+\pi/2}\hat{x}_{\lambda+\pi/2})\right]\right\}\exp\left[p(\xi_\lambda^2 + \xi_{\lambda+\pi/2}^2)/4\right], \tag{3}$$

   where the rotated displacement amplitude is defined as $\xi_\phi \equiv i\left(\xi\exp(-i\phi) - \xi^*\exp(i\phi)\right)/\sqrt{2}$. Show then that the p-ordered, nth moment of the quadrature operator $\hat{x}_\lambda$ can be calculated from the p-ordered characteristic function as,

$$\left\langle \hat{x}_\lambda^n \right\rangle_p = (i\partial/\partial\xi_\lambda)^n \chi(\xi,p)|_{\xi=0}. \tag{4}$$

(d) The quasi-probability distribution $W(\alpha,p)$ over complex α is defined as a two-dimensional Fourier transform of the p-ordered characteristic function of complex ξ,

$$W(\alpha,p) = \frac{1}{\pi^2}\int d^2\xi\,\chi(\xi,p)\exp(\alpha\xi^* - \alpha^*\xi), \tag{5}$$

where $\int d^2\xi \equiv \int_{-\infty}^{\infty} d\xi_r \int_{-\infty}^{\infty} d\xi_i$. Note that $W(\alpha,p)$ is real, but not always positive, hence why we call it a "quasi" probability. Also, this integral is not always well behaved, and $W(\alpha,p)$ must sometimes be expressed in terms of generalized functions (like the Dirac-delta function). The Glauber-Sudarshan (P), Wigner (W), and Husimi (Q) phase-space representations correspond to $p = 1$, 0, and -1, respectively. These phase-space representations can be used as quasi-probability distributions in the sense that the p-ordered expectation (moment) of products of $\hat{a}$ and $\hat{a}^\dagger$ of a quantum optical field can be obtained through integration over α-space of $W(\alpha,p)$,

$$\left\langle \left(\hat{a}^\dagger\right)^m a^n \right\rangle_p = \int d^2\alpha\, W(\alpha,p)(\alpha^*)^m\alpha^n. \tag{6}$$

Show this by directly substituting for $W(\alpha,p)$ in terms of $\chi(\xi,p)$ (in eq. (5)), and using eq. (2). *Hint*: You will need to use the following property of the two-dimensional Dirac-delta function, $\int d^2\xi\, f(\xi)(\partial/\partial\xi)^m(\partial/\partial\xi^*)^n\delta^{(2)}(\xi) = (-\partial/\partial\xi)^m(-\partial/\partial\xi^*)^n f(\xi)|_{\xi=0}$.

(e) Show that $W(\alpha,1) = P(\alpha)$, where $P(\alpha)$ are the elements of the density matrix when diagonalized in the coherent state basis, $\hat{\rho} = \int d^2\alpha P(\alpha)\,|\alpha\rangle\,\langle\alpha|$. Note that the two-dimensional Dirac-delta function can be represented as, $\delta^{(2)}(\xi) \equiv \delta(\xi_r)\delta(\xi_i) = (1/\pi^2)\int d^2\alpha\,\exp(\alpha\xi^* - \alpha^*\xi)$. What is the P-representation for a coherent state $|\alpha_0\rangle$ then? It turns out that all pure states have singular $P(\alpha)$ distributions.

(f) $W(\alpha,0) \equiv W(\alpha)$ is the Wigner distribution. It is a little "smoother" than the P-distribution, but can be negative for certain states (like Fock or number states) so it has its own peculiarities. It can be shown for $p < 1$ (i.e., not for the Glauber-Sudarshan representation) that,

$$W(\alpha, p)|_{p<1} = \frac{2}{\pi(1-p)} \sum_{n=0}^{\infty} \left(\frac{p+1}{p-1}\right)^n \langle n| \hat{D}^\dagger(\alpha) \hat{\rho} \hat{D}(\alpha) |n\rangle . \tag{7}$$

Use this relation to show that the Wigner distribution around $\alpha = 0$ can be used as a sort of photon number parity check of the state of a quantum optical field,

$$W(\alpha = 0) = \frac{2}{\pi} (P_{\text{even}} - P_{\text{odd}}) , \tag{8}$$

where P_{even} is the probability that the field has an even number of photons and P_{odd} is the probability it has an odd number of photons. More generally, $W(\alpha) = (2/\pi)(P_{\text{even}} - P_{\text{odd}})$, where P_{even} and P_{odd} correspond to the probability of even or odd photon parity for the state with density matrix $\hat{\rho}'(\alpha) \equiv \hat{D}^\dagger(\alpha) \hat{\rho} \hat{D}(\alpha)$. Therefore, $|W(\alpha)| \leq 2/\pi$ for all α.

(g) Show using the characteristic function $\chi(\xi, 0) = \text{Tr}[\rho \hat{D}(\xi)]$, that for the vacuum state $\hat{\rho} = |0\rangle\langle 0|$, the Wigner distribution is a Gaussian, $W(\alpha)|_{\hat{\rho}=|0\rangle\langle 0|} = (2/\pi)\exp(-2|\alpha|^2)$. Using the relation $\hat{D}(\alpha)\hat{D}(\alpha') = \hat{D}(\alpha+\alpha')\exp[(\alpha(\alpha')^* - \alpha^*\alpha')/2]$, show for a coherent state of finite amplitude with $\hat{\rho} = |\beta\rangle\langle\beta|$ that the Wigner distribution is simply a displaced version of its vacuum distribution, $W(\alpha)_{\hat{\rho}=|\beta\rangle\langle\beta|} = W(\alpha-\beta)_{\hat{\rho}=|0\rangle\langle 0|}$ (note: this is true generally for any of the p-ordered phase-space representations).

(h) Show that the Wigner distribution can be written in the quadrature basis as,

$$W(x_\lambda, x_{\lambda+\pi/2}) = \frac{1}{2\pi^2} \int d\xi_\lambda \int d\xi_{\lambda+\pi/2} \exp[i(x_\lambda \xi_\lambda + x_{\lambda+\pi/2}\xi_{\lambda+\pi/2})] \times \tag{9}$$

$$\text{Tr}\left\{\hat{\rho}\exp[-i(\hat{x}_\lambda \xi_\lambda + \hat{x}_{\lambda+\pi/2}\xi_{\lambda+\pi/2})]\right\} , \tag{10}$$

where quadrature coordinate x_ϕ is defined as $x_\phi = (1/\sqrt{2})(\alpha\exp(-i\phi) + \alpha^*\exp(i\phi))$ and Fourier quadrature coordinate ξ_ϕ is as defined above. The Wigner distribution can thus be viewed as a joint distribution over x_λ and $x_{\lambda+\pi/2}$. Show that $(1/2)\int dx_{\lambda+\pi/2} W(x_\lambda, x_{\lambda+\pi/2}) = \langle x_\lambda|\hat{\rho}|x_\lambda\rangle$, i.e., it is the (true) probability distribution for the quadrature coordinate x_λ.

(i) Show that the Q-function is equal to, $W(\alpha, -1) \equiv Q(\alpha) = (1/\pi)\langle\alpha|\hat{\rho}|\alpha\rangle$. Solve for the Q-function of a number (Fock) state and a coherent state.

(j) Bonus (5 points): solve for the P, W, and Q representations of a thermal state, $\hat{\rho} = \exp(-\hbar\omega\hat{n}/k_B T)/(\text{Tr}[\exp(-\hbar\hat{n}/k_B T)])$.

[1] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics*, Oxford Series in Optical Imaging and Sciences (Oxford University Press Inc., New York, NY, 1997).

[2] H. J. Carmichael, *Statistical Methods in Quantum Optics 1: Master Equations and Focker-Planck Equations*, Texts and Monographs in Physics (Springer-Verlag, New York, NY, 1999).