Applied Physics 190c: Homework #4

(Dated: May 4, 2017)

Due: Friday, May 12th

1. **Reading** Section III and Appendix E of Clerk2010 [1]. You might also be interested in reading the paper by Schuster, et al. [2], which gives some insight into how one can realize in practice the parametric coupling of a two-level system ("atom") to a resonant cavity/circuit we study theoretically below. In particular, what parameters can be used to set the strength of the coupling. Please note to get full credit for the problems in this set you must show steps in your derivation. It is not good enough to just state the answer!

2. **(30 points) Weak (continuous) position measurement of a mirror**

   Consider the example discussed in class in which a linearly polarized, coherent laser beam is used to measure the position of a mirror by monitoring the phase of the reflected beam.

   (a) Draw a schematic of the set-up, including the required optical components (polarizing beam splitter, waveplate, balanced homodyne detector), that one would use to perform such a quantum-limited measurement of the mirror's position.

   (b) Assume a coherent state laser beam input field, $\tilde{\hat{a}}_{\text{in}} = \tilde{\alpha} + \delta\tilde{\hat{a}}$, where the $\tilde{}$ indicates the slowly varying components of the field (i.e., removing the highly oscillatory optical frequency ω_L), $\tilde{\alpha}$ is a *real* constant amplitude, and $\delta\tilde{\hat{a}}$ is a (continuum) vacuum field input around ω_l. Show that a balanced homodyne (BH) photocurrent measurement of the reflected optical field can be used to estimate the phase of the reflected field through,

$$\hat{\Theta} \equiv \frac{\hat{I}_{-,\phi_{\text{LO}}=0}}{\langle \hat{I}_{-,\phi_{\text{LO}}=-\pi/2} \rangle} = \frac{-i(\tilde{\hat{a}}_r - \tilde{\hat{a}}_r^\dagger)}{2\tilde{\alpha}} \approx \hat{\theta}_r + \frac{-i(\delta\tilde{\hat{a}} - \delta\tilde{\hat{a}}^\dagger)}{2\tilde{\alpha}}, \quad (1)$$

   where $\tilde{\alpha}^2 = \bar{N}$ is the average photon flux of the probe laser field (not the LO!), $\tilde{\hat{a}}_r \equiv \tilde{\hat{a}}_{\text{in}} e^{i\hat{\theta}_r}$, $\hat{\theta}_r$ is the actual phase of the reflected field, and one can assume that $\langle \hat{\theta}_r \rangle \ll 1$ (i.e., only small phase fluctuation around 0 due to motion of the mirror). Here, $\hat{I}_{-,\phi_{\text{LO}}}$ is the BH photocurrent for a local oscillator frequency $\omega_{\text{LO}} = \omega_l$ and phase ϕ_{LO}. We have also assumed a mirror (power) reflectivity of unity.

(c) For ideal BH detection, and no other optical losses, what is the imprecision noise power spectral density $S^I_{xx}(\omega)$ of this measurement of the fluctuations in the position of the mirror? Hint: $\hat{\theta}_r = 2k\hat{x}$, where $k \equiv \omega_l/c$ is the wavevector of the incident laser beam and $\hat{x}$ is the displacement operator of the mirror around its equilibrium point.

(d) Derive the back-action force noise power spectral density $S_{F_xF_x}(\omega)$ of this measurement of the mirror position? What is the product $S^I_{xx}(\omega)S_{F_xF_x}(\omega)$?

(e) Repeat (c) and (d) in the case where the BH detector has a non-unity quantum efficiency of $\eta_{\mathrm{QE}} < 1$.

3. **(50 points) Weak (continuous) measurement of the state of a two-level system using a parametrically coupled resonant cavity**

Consider here the case of a two-level system (TLS) coupled parametrically to a resonant optical cavity. Assume the TLS has a bare transition frequency ω_{01}, the cavity has a bare frequency ω_c, and the interaction Hamiltonian is given by,

$$\hat{H}_{\mathrm{int}} = \hbar\omega_c A \hat{\sigma}_z \hat{a}^\dagger \hat{a}, \tag{2}$$

where A sets the strength of the interaction between probe cavity and the TLS, $\hat{\sigma}_z$ is the Pauli spin operator on the TLS (ground state eigenvalue -1, excited state eigenvalue +1), and $\hat{a}^\dagger\hat{a}$ is the number operator for the number of photons in the resonant optical cavity.

(a) Draw a schematic of the (TLS+cavity+environment)+(optics+BH detector) that could be used to perform a continuous measurement of the state of the TLS using the reflection of a coherent laser field (the "probe" field) from the optical cavity. Assume that the cavity is one-sided such that $\kappa = \kappa_e$, where κ_e is the coupling rate through the front mirror and κ is the total energy decay rate of the cavity. This assumes that the back mirror is perfectly reflective and their are no other cavity loss channels.

(b) For resonant optical driving of the cavity ($\omega_l = \omega_c$), derive a relation for the reflection coefficient $r(\omega_l = \omega_c)$ of the optical field in terms of the TLS operator σ_z.

(c) In the weak measurement limit ($AQ_c \ll 1$, $Q_c = \omega_c/\kappa$ the quality factor of the cavity) show that the phase of the reflected optical field (θ_r) from the cavity is directly related to the state of the TLS through the relation $\hat{\theta}_r \approx 4AQ_c\hat{\sigma}_z$.

(d) If we use a BH detection set-up as in Problem 1, show that the measurement imprecision of the state of the TLS is,

$$S^{I}_{\sigma_z\sigma_z} = \left(\frac{1}{4\bar{\dot{N}}}\right)\left(\frac{1}{\theta_0^2}\right), \tag{3}$$

where $\theta_0 = 4AQ_c$ and $\bar{\dot{N}}$ is the average photon flux of the on-resonant optical probe field.

(e) Show that the noise-power spectral density of the back-action "force" on the TLS due to the measurement probe field is given by $S_{F_{\sigma_z}F_{\sigma_z}} = (\hbar A\omega_c)^2 S_{nn}$, where $S_{nn}(\omega)$ is the intra-cavity photon number noise power spectral density corresponding to the cavity filtered shot-noise of the probe beam.

(f) Derive the general expression for the intra-cavity shot-noise,

$$S_{nn}(\omega) = \bar{n}\left[\frac{\kappa_e}{(\omega+\Delta)^2+(\kappa/2)^2}\right], \tag{4}$$

where the laser-cavity detuning of the probe field is $\Delta \equiv \omega_l - \omega_c$, κ_e is the (energy) coupling rate of the probe field into the optical cavity, κ is the total (energy) decay rate of the cavity field which includes other loss channels besides the input/output coupling port of the probe, and $\bar{n}$ is the average intra-cavity photon number. Show that $\bar{n} = \bar{\dot{N}}\kappa_e/[\Delta^2+(\kappa/2)^2]$ where $\bar{\dot{N}}$ is again the average probe field photon flux.

(g) In the low frequency ($\omega \ll \kappa$), single-sided cavity ($\kappa_e = \kappa$) limit, show that the proposed measurement is a quantum-limited measurement of the TLS state (i.e., $S_{F_{\sigma_z}F_{\sigma_z}} S^{I}_{\sigma_z\sigma_z} = (\hbar/2)^2$).

(h) Under what conditions (if any) is the above measurement a quantum non-demolition (QND) measurement of the state of the TLS?

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[1] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, arXiv:0810.4729 (2010).

[2] D. I. Schuster, A. Wallraff, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. Lett. **94**, 123602 (2005).