Applied Physics 190c: Homework #5

(Dated: May 11, 2017)

Due: Friday, May 19th

1. **Reading:** Section IV and Appendix I of Clerk2010 [1].

2. **(60 points) Quantum Noise Contraint on a Linear Detector**

As in Section IV and Appendix I of Clerk2010 [1], assume a linear detector with input observable $\hat{F}$ coupled to signal $\hat{x}$ and output observable $\hat{I}$ coupled to load $\hat{y}$. The interaction at the detector input is defined by $\hat{H}_{\text{int}}^{\text{in}} = A\hat{x}\hat{F}$ and at its output by $\hat{H}_{\text{int}}^{\text{out}} = B\hat{y}\hat{I}$. We proposed in class that such a linear detector had a quantum constraint on its noise power spectral densities (NPSDs),

$$\bar{S}_{II}[\omega]\bar{S}_{FF}[\omega] - \left|\bar{S}_{IF}[\omega]\right|^2 \geq \left|\frac{\hbar\tilde{\chi}_{IF}[\omega]}{2}\right|^2 \left(1 + \Delta\left[\frac{\bar{S}_{IF}[\omega]}{\hbar\tilde{\chi}_{IF}[\omega]/2}\right]\right), \tag{1}$$

where $\chi_{IF}[\omega]$ ($\chi_{FI}[\omega]$) is the Fourier transform of the forward (reverse) gain response function (susceptibility) of the detector, $\tilde{\chi}_{IF}[\omega] \equiv \chi_{IF}[\omega] - (\chi_{FI}[\omega])^*$, and $\Delta[z] \equiv \left[\left|1+z^2\right| - (1+|z|^2)\right]/2$. Note that we are using the usual definition of symmetrized NPSDs as per Ref. [1], and that we have removed the average values from our detector input and output observables in the absence of coupling to the source or load.

(a) Show using first-order perturbation theory (making the required assumptions for this to be a valid approximation) that the average linear response of the output observable of the detector ($\hat{I}$) to an input signal ($\hat{x}$) is,

$$\langle\hat{I}(t)\rangle = \langle\hat{I}\rangle_0 + A\int dt'\chi_{IF}(t-t')\langle\hat{x}(t')\rangle, \tag{2}$$

with the forward detector susceptibility given by,

$$\chi_{IF}(t) = -\frac{i}{\hbar}\theta(t)\left\langle\left[\hat{I}(t), \hat{F}(0)\right]\right\rangle_0. \tag{3}$$

Note that here and in what follows $\hat{I}$ and $\hat{F}$ are Heisenberg operators with respect to the uncoupled detector Hamiltonian such that the free time evolution of the detector is contained completely in the observable operators, and $\langle\rangle_0$ indicates an expectation value against the density operator of the uncoupled detector.

(b) What is the similar relation for the reverse gain system response?

(c) From the definition of the unsymmetrized I-F noise correlator $S_{IF}[\omega]$ show that,

$$\bar{S}_{IF}[\omega] = \left[S_{IF}[\omega] + (S_{IF}[-\omega])^*\right], \tag{4}$$

and that

$$\tilde{\chi}_{IF}[\omega] = -\frac{i}{\hbar}\left[S_{IF}[\omega] - (S_{IF}[-\omega])^*\right]. \tag{5}$$

(d) For the cosine-transform operators of $\hat{I}(t)$ and $\hat{F}(t)$,

$$\hat{A}[\omega] \equiv \sqrt{\frac{2}{T}}\int_{-T/2}^{T/2} dt\cos(\omega t + \delta)\hat{I}(t), \tag{6}$$

$$\hat{B}[\omega] \equiv \sqrt{\frac{2}{T}}\int_{-T/2}^{T/2} dt\cos(\omega t)\hat{F}(t), \tag{7}$$

show that under the limit $T \to \infty$ and for any finite $\omega \neq 0$ that,

$$(\Delta A)^2 = \bar{S}_{II}[\omega], \tag{8}$$

$$(\Delta B)^2 = \bar{S}_{FF}[\omega], \tag{9}$$

$$\left\langle\{\hat{A},\hat{B}\}\right\rangle = 2\,\mathrm{Re}\left(e^{i\delta}\bar{S}_{IF}[\omega]\right), \tag{10}$$

$$\left\langle[\hat{A},\hat{B}]\right\rangle = i\hbar\,\mathrm{Re}\left(e^{i\delta}\tilde{\chi}_{IF}[\omega]\right). \tag{11}$$

(e) Derive the generalized form of the Heisenberg uncertainty relation for two observables (with zero means) in terms of their commutator and their noise correlator,

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left[\left\langle\{\hat{A},\hat{B}\}\right\rangle^2 + \left|\left\langle[\hat{A},\hat{B}]\right\rangle\right|^2\right]. \tag{12}$$

(f) From your results in (d) and (e), prove the generalized quantum noise constraint of eq. (1).

(g) We call a detector that saturates the quantum noise constraint bound of eq. (1) a "quantum-ideal" detector. Saturating this bound puts significant restrictions on the relationship between $S_{II}[\omega]$, $S_{FF}[\omega]$, and $S_{IF}[\omega]$. Refering to Appendix I of Ref. [1], state (without proof) the series of relationships that these NPSDs must satisfy for a "quantum-ideal" detector. If the detector is also to have non-vanishing gain and power gain, what constraint does this further put on the properties of the detector in terms of the detector final states that contribute to the detector noise?

[1] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, arXiv:0810.4729 (2010).